

SOLVABILITY OF A NONLINEAR DIFFERENCE EQUATION OF THE FIFTEENTH ORDER

STEVO STEVIĆ*, BRATISLAV IRIČANIN AND WITOLD KOSMALA

Abstract. We find some closed-form formulas for the general solution to the difference equation

$$x_{n+1} = \frac{x_{n-2}x_{n-8}x_{n-14}}{x_{n-5}x_{n-11}(a + bx_{n-2}x_{n-8}x_{n-14})}, \quad n \in \mathbb{N}_0,$$

where $a, b \in \mathbb{C}$, $a^2 + b^2 \neq 0$, and $x_{-j} \in \mathbb{C} \setminus \{0\}$, $j = \overline{0, 14}$, explaining the formulas for the four special cases of the difference equation which have appeared quite recently in the literature. We also give several comments on the claims on the behaviour of solutions to the four difference equations given therein, as well as some counterexamples.

Mathematics subject classification (2020): Primary 39A20, 39A45.

Keywords and phrases: Nonlinear difference equation, solvable difference equation, closed-form formula.

REFERENCES

- [1] D. ADAMOVIĆ, *Solution to problem 194*, Mat. Vesnik, **23** (1971), 236–242.
- [2] I. BAJO AND E. LIZ, *Global behaviour of a second-order nonlinear difference equation*, J. Differ. Equat. Appl. **17** (10) (2011), 1471–1486.
- [3] L. BERG AND S. STEVIĆ, *On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$* , J. Difference Equ. Appl. **17** (4) (2011), 577–586.
- [4] D. BERNOULLI, *Observationes de seriebus quae formantur ex additione vel subtractione quacunq̄ue terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur*, Commentarii Acad. Petropol. III, 1728 (1732), 85–100, (in Latin).
- [5] G. BOOLE, *A Treatise on the Calculus of Finite Differences*, Third Edition, Macmillan and Co., London, 1880.
- [6] B. P. DEMIDOVICH AND I. A. MARON, *Computational Mathematics*, Mir Publishers, Moscow, 1973.
- [7] A. DE MOIVRE, *Miscellanea analytica de seriebus et quadraturis*, Londini, 1730, (in Latin).
- [8] L. EULER, *Introductio in Analysin Infinitorum, Tomus Primus*, Lausannae, 1748, (in Latin).
- [9] T. FORT, *Finite Differences and Difference Equations in the Real Domain*, Oxford Univ. Press, London, 1948.
- [10] J. F. W. HERSCHEL, *Collection of Examples of the Applications of the Calculus of Finite Differences*, J. Smith, Cambridge, 1820.
- [11] B. IRIČANIN, S. STEVIĆ, *On some rational difference equations*, Ars Combin. **92** (2009), 67–72.
- [12] C. JORDAN, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1965.
- [13] G. KARAKOSTAS, *The forbidden set, solvability and stability of a circular system of complex Riccati type difference equations*, AIMS Mathematics **8** (11) (2023), 28033–28050.

- [14] V. A. KRECHMAR, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.
- [15] S. F. LACROIX, *Traité des Différences et des Séries*, J. B. M. Duprat, Paris, 1800, (in French).
- [16] S. F. LACROIX, *An Elementary Treatise on the Differential and Integral Calculus, with an Appendix and Notes by J. Herschel, J. Smith*, Cambridge, 1816.
- [17] J.-L. LAGRANGE, *Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes*, Miscellanea Taurinensia, t. I, (1759), 33–42 (Lagrange OEuvres, I, 23–36, 1867) (in French).
- [18] J.-L. LAGRANGE, *OEuvres*, t. II, Gauthier-Villars, Paris, 1868, (in French).
- [19] P. S. LAPLACE, *Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards*, Mémoires de l'Académie Royale des Sciences de Paris 1773, t. VII, (1776) (Laplace OEuvres, VIII, 69–197, 1891), (in French).
- [20] P. S. LAPLACE, *Sur le calcul des fonctions génératrices*, Journal de l'École Polytechnique, Tome VIII, Cah. 15, (1810), 229–265, (in French).
- [21] A. A. MARKOFF, *Differenzenrechnung*, Teubner, Leipzig, 1896, (in German).
- [22] L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, MacMillan and Co., London, 1933.
- [23] D. S. MITRINOVIĆ AND J. D. KEČKIĆ, *Metodi Izračunavanja Konačnih Zbirova*, Naučna Knjiga, Beograd, 1984, (in Serbian).
- [24] N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924, (in German).
- [25] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *On a system of two nonlinear difference equations*, J. Math. Anal. Appl. **219** (2) (1998), 415–426.
- [26] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *On the behavior of the solutions of a system of two nonlinear difference equations*, Comm. Appl. Nonlinear Anal. **5** (2) (1998), 47–59.
- [27] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *Invariants for systems of two nonlinear difference equations*, Differ. Equ. Dyn. Syst. **7** (1999), 181–196.
- [28] G. PAPASCHINOPOULOS AND C. J. SCHINAS, *Invariants and oscillation for systems of two nonlinear difference equations*, Nonlinear Anal. Theory Methods Appl. **46** (2001), 967–978.
- [29] G. PAPASCHINOPOULOS, C. J. SCHINAS, AND G. STEFANIDOU, *On a k -order system of Lyness-type difference equations*, Adv. Difference Equ. Vol. 2007, Article ID 31272, (2007), 13 pages.
- [30] G. PAPASCHINOPOULOS, C. J. SCHINAS AND G. STEFANIDOU, *Two modifications of the Beverton-Holt equation*, Int. J. Difference Equ. **4** (1) (2009), 115–136.
- [31] G. PAPASCHINOPOULOS AND G. STEFANIDOU, *Trichotomy of a system of two difference equations*, J. Math. Anal. Appl. **289** (2004), 216–230.
- [32] G. PAPASCHINOPOULOS AND G. STEFANIDOU, *Asymptotic behavior of the solutions of a class of rational difference equations*, Inter. J. Difference Equations **5** (2) (2010), 233–249.
- [33] J. RIORDAN, *Combinatorial Identities*, John Wiley & Sons Inc., New York-London-Sydney, 1968.
- [34] C. SCHINAS, *Invariants for difference equations and systems of difference equations of rational form*, J. Math. Anal. Appl. **216** (1997), 164–179.
- [35] C. SCHINAS, *Invariants for some difference equations*, J. Math. Anal. Appl. **212** (1997), 281–291.
- [36] D. SIMSEK, B. OGUL AND F. ABDULLAYEV, *Dynamical behavior of solution of fifteenth-order rational difference equation*, Filomat **38** (3) (2024), 997–1008.
- [37] S. STEVIĆ, *On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II*, Dyn. Contin. Discrete Impuls. Syst. **10a** (6) (2003), 911–916.
- [38] S. STEVIĆ, *Boundedness character of a class of difference equations*, Nonlinear Anal. TMA **70** (2009), 839–848.
- [39] S. STEVIĆ, *On the difference equation $x_n = x_{n-k}/(b + cx_{n-1} \cdots x_{n-k})$* , Appl. Math. Comput. **218** (2012), 6291–6296.

- [40] S. STEVIĆ, *On the system $x_{n+1} = y_n x_{n-k} / (y_{n-k+1} (a_n + b_n y_n x_{n-k}))$, $y_{n+1} = x_n y_{n-k} / (x_{n-k+1} (c_n + d_n x_n y_{n-k}))$* , Appl. Math. Comput. **219** (2013), 4526–4534.
- [41] S. STEVIĆ, *On the system of difference equations $x_n = c_n y_{n-3} / (a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3} / (\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$* , Appl. Math. Comput. **219** (2013), 4755–4764.
- [42] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *On a third-order system of difference equations with variable coefficients*, Abstr. Appl. Anal. Vol. 2012, Article ID 508523, (2012), 22 pages.
- [43] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *On some solvable difference equations and systems of difference equations*, Abstr. Appl. Anal. Vol. 2012, Article ID 541761, (2012), 11 pages.
- [44] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *On the difference equation $x_n = a_n x_{n-k} / (b_n + c_n x_{n-1} \cdots x_{n-k})$* , Abstr. Appl. Anal. Vol. 2012, Article ID 409237, (2012), 19 pages.
- [45] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *On a solvable system of rational difference equations*, J. Difference Equ. Appl. **20** (5–6) (2014), 811–825.
- [46] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *Solvability of nonlinear difference equations of fourth order*, Electron. J. Differential Equations Vol. 2014, Article No. 264, (2014), 14 pages.
- [47] S. STEVIĆ, J. DIBLIK, B. IRIČANIN AND Z. ŠMARDÁ, *On a fifth-order difference equation*, J. Comput. Anal. Appl. **20** (7) (2016), 1214–1227.
- [48] S. STEVIĆ, B. IRIČANIN, W. KOSMALA AND Z. ŠMARDÁ, *Note on the bilinear difference equation with a delay*, Math. Methods Appl. Sci. **41** (2018), 9349–9360.
- [49] S. STEVIĆ, B. IRIČANIN AND Z. ŠMARDÁ, *On a close to symmetric system of difference equations of second order*, Adv. Difference Equ. Vol. 2015, Article No. 264, (2015), 17 pages.
- [50] S. STEVIĆ, B. IRIČANIN AND Z. ŠMARDÁ, *On a product-type system of difference equations of second order solvable in closed form*, J. Inequal. Appl. Vol. 2015, Article No. 327, (2015), 15 pages.
- [51] N. N. VOROBIEV, *Fibonacci Numbers*, Birkhäuser, Basel, 2002.