

THE CLAUSING INEQUALITY AND STRONG  $\mathcal{F}$ -CONCAVITY

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*Abstract.* In this paper, we introduce the class of strongly  $\mathcal{F}$ -concave functions as the class of functions  $f: I \rightarrow \mathbb{R}$ , where  $I \subseteq \mathbb{R}$  is an interval, which satisfy

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \leq F(tx + (1-t)y) - tF(x) - (1-t)F(y)$$

for  $x, y \in I$  and  $t \in [0, 1]$  and some convex function  $F$  on  $I$  called control function. This class contains the class of strongly concave functions. Analogous generalization of strongly convex functions is also given.

We investigate possibilities to use this class to refine the Clausing inequality. The refinement of the left-hand side of the Clausing inequality has the same form as refinements of any Jensen type inequality (for example, the Hermite-Hadamard inequality), but we introduce a suitable class of control functions  $F$  such that these refinements are applicable to much broader class of  $\mathcal{F}$ -concave functions than it is possible for strongly concave functions. The refinements for the right-hand side of the inequality are more subtle to obtain, but flexibility of choosing control functions enables us to refine this side also.

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