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# INEQUALITIES ON THE ESSENTIAL JOINT AND ESSENTIAL GENERALIZED SPECTRAL RADIUS

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*Abstract.* We prove new inequalities for the essential generalized and the essential joint spectral radius of Hadamard (Schur) weighted geometric means of bounded sets of infinite nonnegative matrices that define operators on suitable Banach sequence spaces and of bounded sets of positive kernel operators on $L^2$. To our knowledge the obtained inequalities are new even in the case of singelton sets.

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Paper JMI-18-85

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