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A NEW APPROACH TO SELECTING CONSTANTS FOR SOME ANALYTIC INEQUALITIES

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Abstract. The subject of the paper is a new approach to selecting real constants for which certain analytic inequalities hold. This approach is based on the introduction and analysis of the corresponding families of functions that are stratified and such that each function from the family has certain Taylor expansions. The approach is illustrated on some D'Aurizio-Sándor-type inequalities that were previously proved only for values of parameters that are natural numbers. In this paper, we analyse and prove those inequalities for all real values of the parameters for which they are defined. Our approach has enabled selecting the best real constants for which those inequalities hold.

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