Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 307–331
doi:10.7153/jmi-2025-19-22

NUMERICAL RADIUS INEQUALITIES FOR THE WEIGHTED SUMS OF HILBERT SPACE OPERATORS

SONGYUE PANG AND YUXIA LIANG*

Abstract. Using the generalized Young inequality, operator convexity and positive operator matrix, we extend and refine some numerical radius inequalities for the weighted sums of Hilbert space operators. Precisely, for $r,t\in[1,2]$, if $T_k,V_k\in\mathscr{B}(\mathscr{H})$ $(k=1,\dots,n)$ and $p_k\geqslant 0$ with $\sum_{k=1}^n p_k=P_n$, then

$$w^{2r}\left(\frac{1}{P_n}\sum_{k=1}^n p_kT_kV_k\right)\leqslant\frac{1}{2^{1/t}}w^{\frac{2}{t}}\left(\frac{1}{P_n}\sum_{k=1}^n p_k\left(|T_k^*|^{2rt}+i|V_k|^{2rt}\right)\right)-\inf_{\|x\|=1}\phi(x),$$

where

$$\phi(x)=\frac{1}{4}\left(\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|T_k^*|^{2r}\right)x,x\right\rangle-\left\langle\left(\frac{1}{P_n}\sum_{k=1}^n p_k|V_k|^{2r}\right)x,x\right\rangle\right)^2.$$

Mathematics subject classification (2020): 47A12, 47A30, 15A60.
Keywords and phrases: Numerical radius, convex function operator, Young inequality.

REFERENCES

[1] Y. AL-MANASRAH, F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity. **19** (2015) 757–768.
[2] T. ANDO, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. **26** (1979) 203–241.
[3] M. L. BUZANO, *Generalizzazione della diseguaglianza di Cauchy-Schwarz*, Rend. Sem. Mat. Univ. e Politech. Torino. **31** (1974) 405–409.
[4] P. BHUNIA, S. JANA, K. PAUL, *Numerical radius inequalities and estimation of zeros of polynomials*, Georgian Math. J. **30** (5) (2023) 671–682.
[5] P. BHUNIA, *Improved bounds for the numerical radius via polar decomposition of operators*, Linear Algebra Appl. **683** (2024) 31–45.
[6] P. BHUNIA, S. S. DRAGOMIR, M. S. MOSLEHIAN et al., *Lectures on numerical radius inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer Nature, 2022.
[7] P. BHUNIA, K. PAUL, *New upper bounds for the numerical radius of Hilbert space operators*, Bull. Sci. Math. **167** (2021) 102959.
[8] S. S. DRAGOMIR, *Power inequalities for the numerical radius of a product of two operators in Hilbert spaces*, Research report collection. **11** (4) (2008).
[9] S. S. DRAGOMIR, *Inequalities for the numerical radius of linear operators in Hilbert spaces*, Cham: Springer, 2013.
[10] M. EL-HADDAD, F. KITTANEH, *Numerical radius inequalities for Hilbert space operators. II*, Studia Math. **182** (2007) 133–140.
[11] T. FURUTA, *Invitation to linear operators: From matrices to bounded linear operators on a Hilbert space*, CRC Press, 2001.
[12] K. E. GUSTAFSON, D. K. M. RAO, K. E. GUSTAFSON et al., *Numerical range. The Field of Values of Linear Operators and Matrices*, Springer New York, 1997.

[13] Z. HEYDARBEYGI, M. SABABHEH, H. MORADI, *A convex treatment of numerical radius inequalities*, Czechoslovak Math. J. **72** (2) (2022) 601–614.
[14] F. KITTANEH, H. R. MORADI, M. SABABHEH, *Sharper bounds for the numerical radius*, Linear Multilinear Algebra. (2023) 1–11.
[15] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Stud. Math. **168** (1) (2005) 73–80.
[16] F. KITTANEH, M. S. MOSLEHIAN, T. YAMAZAKI, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015) 46–53.
[17] F. KITTANEH, M. S. MOSLEHIAN, T. YAMAZAKI, *Cartesian decomposition and numerical radius inequalities*, Linear Algebra Appl. **471** (2015) 46–53.
[18] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (2) (1988) 283–293.
[19] H. R. MORADI, M. SABABHEH, *New estimates for the numerical radius*, arxiv preprint arxiv:2010.12756, 2020.
[20] M. H. M. RASHID, N. H. ALTAWEEL, *Some generalized numerical radius inequalities for Hilbert space operators*, J. Math. Inequal. **16** (2) (2022) 541–560.
[21] T. M. RASSIAS, *Survey on Classical Inequalities*, Kluwer Academic Publishers, Dordrecht. 2000.
[22] B. SIMON, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1979.
[23] X. H. WU, J. J. HUANG, A. CHEN, *On the origin and numerical range of bounded operators*, Acta Math. Sin. **35** (10) (2019) 1715–1722.