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# ON SEQUENCES SATISFYING FOURTH–ORDER DIFFERENCE INEQUALITIES

BESSEM SAMET

*Abstract.* We consider the class of sequences $a=(a_1,\cdots,a_n)\in\mathbb{R}^n$ satisfying the fourth-order difference inequality $\Delta^4 a_i\leqslant 0$, $i=1,\cdots,n-4$. A Hermite-Hadamard-type inequality is established for this class of sequences. The proof of our result is based on the choice of an appropriate sequence which is the solution to a certain fourth-order difference equation. Moreover, if $a$ is a convex sequence, we obtain an interesting refinement of the right discrete Hermite-Hadamard inequality. We next extend our study to the class of matrices satisfying a system of fourth-order difference inequalities. In particular, we obtain a trace inequality for the class of symmetric matrices.

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