Journal of Mathematical Inequalities
Volume 19, Number 1 (2025), 387–419
doi:10.7153/jmi-2025-19-26

FRACTIONAL INTEGRAL APPROACH TO PARAMETERIZED INEQUALITIES FOR (s,P)–PREINVEXITY

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Abstract. Fractional calculus is an invaluable tool with significant potential for application in the physical sciences. This paper focuses on addressing parameterized fractional inequalities for (s,P)-preinvex functions. In light of this, we introduce the concept of (s,P)-preinvex functions and investigate their related properties. By considering limited first- and second-order derivative functions, we present two fractional integral identities with a single parameter using exponential kernel fractional integrals. Building upon these identities, we establish parameterized integral inequalities for (s,P)-preinvex functions. To provide a more intuitive display of the results, we also offer illustrative examples with graphs to demonstrate the validity of our theoretical findings.

Mathematics subject classification (2020): 26A33, 26A51, 41A55, 26D15.
Keywords and phrases: Hermite–Hadamard-type inequalities, fractional integrals, (s,P)-preinvex functions.

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Paper JMI-19-26

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