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THE SKEW GENERALIZED VON NEUMANN–JORDAN CONSTANT IN THE UNIT SPHERE

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Abstract. In this paper, we introduce a new constant for Banach spaces, denoted as $\widetilde{C}_{\mathrm{NJ}}^{p}(\xi,\nu,X)$. We provide calculations for both the lower and upper bounds of this constant, as well as its exact values in certain Banach spaces. Furthermore, we give the inequality relationship between the $\widetilde{C}_{\mathrm{NJ}}^{p}(\xi,\nu,X)$ constant and the other two constants. Besides, we establish an equivalent relationship between the $\widetilde{C}_{\mathrm{NJ}}^{p}(\xi,\nu,X)$ constant and the $\widetilde{C}_{\mathrm{NJ}}^{(p)}(X)$ constant. Specifically, we shall exhibit the connections between the constant $\widetilde{C}_{\mathrm{NJ}}^{p}(\xi,\nu,X)$ and certain geometric characteristics of Banach spaces, including uniform convexity and uniform nonsquareness. Additionally, a sufficient condition for normal structure about the $\widetilde{C}_{\mathrm{NJ}}^{p}(\xi,\nu,X)$ constant is also established.

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