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ABSOLUTE MONOTONICITY OF FOUR FUNCTIONS INVOLVING THE SECOND KIND OF COMPLETE ELLIPTIC INTEGRALS

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Abstract. In the study, the authors present absolute monotonicity of four functions involving the inverse hyperbolic tangent function and the second kind of complete elliptic integrals, derive four double inequalities for bounding the second kind of complete elliptic integrals, and acquire an upper bound of the Hersch–Pfluger distortion function. These inequalities improve several known ones. Moreover, the authors connect two of the four functions with normalized remainders of the Maclaurin series of two functions involving the inverse hyperbolic tangent function and the second kind of complete elliptic integrals.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] R. P. AGARWAL, E. KARAPINAR, M. KOSTIĆ, J. CAO, AND W.-S. DU, *A brief overview and survey of the scientific work by Feng Qi*, Axioms **11** (2022), no. 8, Paper No. 385, 27 pages; available online https://doi.org/10.3390/axioms11080385.

[3] H. ALZER AND S.-L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math. **172** (2004), no. 2, 289–312; available online at https://doi.org/10.1016/j.cam.2004.02.009.

[4] H. ALZER AND K. RICHARDS, *A note on a function involving complete elliptic integrals: monotonicity, convexity, inequalities*, Anal. Math. **41** (2015), no. 3, 133–139; available online at https://doi.org/10.1007/s10476-015-0201-7.

[5] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[6] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, Cambridge, 1999; available online at https://doi.org/10.1017/CBO9781107325937.

[7] Z.-H. BAO, R. P. AGARWAL, F. QI, AND W.-S. DU, *Some properties on normalized tails of Maclaurin power series expansion of exponential function*, Symmetry **16** (2024), no. 8, Art. 989, 15 pages; available online at https://doi.org/10.3390/sym16080989.

[8] C. BERG, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439; available online at https://doi.org/10.1007/s00009-004-0022-6.

[9] L. CARLITZ, *The Staudt–Clausen theorem*, Math. Mag. **34** (1960/61), 131–146; available online at https://doi.org/10.2307/2688488.

[10] W.-S. DU, R. P. AGARWAL, E. KARAPINAR, M. KOSTIĆ, AND J. CAO, *Preface to the Special Issue "A Themed Issue on Mathematical Inequalities, Analytic Combinatorics and Related Topics in Honor of Professor Feng Qi"*, Axioms **12** (2023), no. 9, Art. 846, 5 pages; available online at https://doi.org/10.3390/axioms12090846.

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[11] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at https://doi.org/10.1016/B978-0-12-384933-5.00013-8.

[12] B.-N. GUO AND F. QI, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.

[13] B.-N. GUO AND F. QI, *Some bounds for the complete elliptic integrals of the first and second kind*, Math. Inequal. Appl. **14** (2011), no. 2, 323–334; available online at https://doi.org/10.7153/mia-14-26.

[14] F. T. HOWARD, *A special class of Bell polynomials*, Math. Comp. **35** (1980), no. 151, 977–989; available online at https://doi.org/10.2307/2006208.

[15] F. T. HOWARD, *A sequence of numbers related to the exponential function*, Duke Math. J. **34** (1967), 599–615; available online at http://projecteuclid.org/euclid.dmj/1077377163.

[16] F. T. HOWARD, *A sequence of numbers related to the exponential function*, Thesis (Ph.D.)-Duke University. ProQuest LLC, Ann Arbor, MI, 1966. 48 pp. available online at http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:6706872.

[17] F. T. HOWARD, *Associated Stirling numbers*, Fibonacci Quart. **18** (1980), no. 4, 303–315.

[18] F. T. HOWARD, *Numbers generated by the reciprocal of* $e^x - x - 1$, Math. Comp. **31** (1977), no. 138, 581–598; available online at https://doi.org/10.2307/2006440.

[19] F. T. HOWARD, *Some sequences of rational numbers related to the exponential function*, Duke Math. J. **34** (1967), 701–716; available online at http://projecteuclid.org/euclid.dmj/1077377305.

[20] O. HÜBNER, *Remarks on a paper by Ławrynowicz on quasiconformal mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18** (1970), 183–186.

[21] O. LEHTO AND K. I. VIRTANEN, *Quasiconformal Mappings in the Plane*, second edition, Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[22] Y.-W. LI AND F. QI, *A new closed-form formula of the Gauss hypergeometric function at specific arguments*, Axioms **13** (2024), no. 5, Art. 317, 24 pages; available online at https://doi.org/10.3390/axioms13050317.

[23] Y.-F. LI AND F. QI, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing cosine*, Open Math. **21** (2023), no. 1, Paper No. 20230159, 12 pages; available online at https://doi.org/10.1515/math-2023-0159.

[24] Y.-W. LI, F. QI, AND W.-S. DU, *Two forms for Maclaurin power series expansion of logarithmic expression involving tangent function*, Symmetry **15** (2023), no. 9, Art. 1686, 18 pages; available online at https://doi.org/10.3390/sym15091686.

[25] X.-L. LIU, H.-X. LONG, AND F. QI, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing sine*, Mathematics **11** (2023), no. 14, Art. 3107, 12 pages; available online at https://doi.org/10.3390/math11143107.

[26] X.-L. LIU AND F. QI, *Monotonicity results of ratio between two normalized remainders of Maclaurin series expansion for square of tangent function*, Math. Slovaca **75** (2025), no. 3, 699–705; available online at https://doi.org/10.1515/ms-2025-0051.

[27] X. MA, Y. CHU, AND F. WANG, *Monotonicity and inequalities for the generalized distortion function*, Acta Math. Sci. Ser. B (Engl. Ed.) **33** (2013), no. 6, 1759–1766; available online at https://doi.org/10.1016/S0252-9602(13)60121-6.

[28] T. MANSOUR AND M. SCHORK, *Commutation Relations, Normal Ordering, and Stirling Numbers*, Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 2016.

[29] G. V. MILOVANOVIĆ AND F. QI, *Closed-form formulas of two Gauss hypergeometric functions of specific parameters*, J. Math. Anal. Appl. **543** (2025), no. 2, part 3, Paper No. 129024, 35 pages; available online at https://doi.org/10.1016/j.jmaa.2024.129024.

[30] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at https://doi.org/10.1007/978-94-017-1043-5.

[31] D.-W. NIU AND F. QI, *Monotonicity results of ratios between normalized tails of Maclaurin power series expansions of sine and cosine*, Mathematics **12** (2024), no. 12, Art. 1781, 20 pages; available online at https://doi.org/10.3390/math12121781.

[32] W.-J. PEI AND B.-N. GUO, *Monotonicity, convexity, and Maclaurin series expansion of Qi's normalized remainder of Maclaurin series expansion with relation to cosine*, Open Math. **22** (2024), no. 1, Paper No. 20240095, 11 pages; available online at https://doi.org/10.1515/math-2024-0095.

[33] G. PÓLYA AND G. SZEGÖ, *Problems and Theorems in Analysis*, Volume I, *Series, integral calculus, theory of functions*, Translated from the German by D. Aeppli. Corrected printing of the revised translation of the fourth German edition. Grundlehren der Mathematischen Wissenschaften, 193. Springer-Verlag, Berlin-New York, 1978.

[34] A. P. PRUDNIKOV, YU. A. BRYCHKOV, AND O. I. MARICHEV, *Integrals and Series*, Vol. 3. *More Special Functions*. Translated from the Russian by G. G. Gould. Gordon and Breach Science Publishers, New York, 1990.

[35] F. QI, *Absolute monotonicity of normalized tail of power series expansion of exponential function*, Mathematics **12** (2024), no. 18, Art. 2859, 11 pages; available online at https://doi.org/10.3390/math12182859.

[36] F. QI, *Series and connections among central factorial numbers, Stirling numbers, inverse of Vandermonde matrix, and normalized remainders of Maclaurin series expansions*, Mathematics **13** (2025), no. 2, Art. 223, 52 pages; available online at https://doi.org/10.3390/math13020223.

[37] F. QI, R. P. AGARWAL, AND D. LIM, *Decreasing property of ratio of two logarithmic expressions involving tangent function*, Math. Comput. Model. Dyn. Syst. **31** (2025), no. 1, 1–16; available online at https://doi.org/10.1080/13873954.2024.2449322.

[38] F. QI AND C.-P. CHEN, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), 603–607; available online at https://doi.org/10.1016/j.jmaa.2004.04.026.

[39] F. QI, L.-H. CUI, AND S.-L. XU, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528; available online at https://doi.org/10.7153/mia-02-42.

[40] F. QI AND C.-J. HUANG, *Computing sums in terms of beta, polygamma, and Gauss hypergeometric functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), Art. 191, 9 pages; available online at https://doi.org/10.1007/s13398-020-00927-y.

[41] F. QI AND Z. HUANG, *Inequalities of the complete elliptic integrals*, Tamkang J. Math. **29** (1998), no. 3, 165–169; available online at https://doi.org/10.5556/j.tkjm.29.1998.4242.

[42] F. QI, D.-W. NIU, D. LIM, AND Y.-H. YAO, *Special values of the Bell polynomials of the second kind for some sequences and functions*, J. Math. Anal. Appl. **491** (2020), no. 2, Art. 124382, 31 pages; available online at https://doi.org/10.1016/j.jmaa.2020.124382.

[43] S.-L. QIU, X.-Y. MA, AND Y.-M. CHU, *Sharp Landen transformation inequalities for hypergeometric functions, with applications*, J. Math. Anal. Appl. **474** (2019), no. 2, 1306–1337; available online at https://doi.org/10.1016/j.jmaa.2019.02.018.

[44] S.-L. QIU, X.-Y. MA, AND Y.-M. CHU, *Transformation properties of hypergeometric functions and their applications*, Comput. Methods Funct. Theory **22** (2022), no. 2, 323–366; available online at https://doi.org/10.1007/s40315-021-00390-9.

[45] S.-L. QIU AND M. K. VAMANAMURTHY, *Elliptic integrals and the modulus of Grötzsch ring*, PanAmer. Math. J. **5** (1995), no. 2, 41–60.

[46] S.-L. QIU AND M. VUORINEN, *Infinite products and normalized quotients of hypergeometric functions*, SIAM J. Math. Anal. **30** (1999), no. 5, 1057–1075; available online at https://doi.org/10.1137/S0036141097326805.

[47] J. QUAINTANCE AND H. W. GOULD, *Combinatorial Identities for Stirling Numbers*, The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

[48] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at https://doi.org/10.1515/9783110269338.

[49] N. M. TEMME, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at https://doi.org/10.1002/9781118032572.

[50] A. WAN AND F. QI, *Power series expansion, decreasing property, and concavity related to logarithm of normalized tail of power series expansion of cosine*, Electron. Res. Arch. **32** (2024), no. 5, 3130–3144; available online at https://doi.org/10.3934/era.2024143.

[51] M. K. WANG, Y. M. CHU, Y. P. JIANG, AND D. D. YAN, *A class of quadratic transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci. Ser. A (Chinese Ed.) **34** (2014), no. 4, 999–1007. (Chinese)

[52] M.-K. WANG, Y.-M. CHU, AND S.-L. QIU, *Sharp bounds for generalized elliptic integrals of the first kind*, J. Math. Anal. Appl. **429** (2015), no. 2, 744–757; available online at https://doi.org/10.1016/j.jmaa.2015.04.035.

[53] M.-K. WANG, Y.-M. CHU, AND Y.-Q. SONG, *Asymptotical formulas for Gaussian and generalized hypergeometric functions*, Appl. Math. Comput. **276** (2016), 44–60; available online at https://doi.org/10.1016/j.amc.2015.11.088.

[54] M.-K. WANG, S.-L. QIU, Y.-M. CHU, AND Y.-P. JIANG, *Generalized Hersch-Pfluger distortion function and complete elliptic integrals*, J. Math. Anal. Appl. **385** (2012), no. 1, 221–229; available online at https://doi.org/10.1016/j.jmaa.2011.06.039.

[55] F. WANG, B.-N. GUO, AND F. QI, *Monotonicity and inequalities related to complete elliptic integrals of the second kind*, AIMS Math. **5** (2020), no. 3, 2732–2742; available online at https://doi.org/10.3934/math.2020176.

[56] F. WANG AND F. QI, *Monotonicity and sharp inequalities related to complete (p,q)-elliptic integrals of the first kind*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 8, 961–970; available online at https://doi.org/10.5802/crmath.119.

[57] F. WANG AND F. QI, *Power series expansion and decreasing property related to normalized remainders of power series expansion of sine*, Filomat **38** (2024), no. 29, 10447–10462; available online at https://doi.org/10.2298/FIL2429447W.

[58] G. D. WANG, X. H. ZHANG, Y. M. CHU, AND S. L. QIU, *Complete elliptic integrals and the Hersch–Pfluger distortion function*, Acta Math. Sci. Ser. A (Chinese Ed.) **28** (2008), no. 4, 731–734. (Chinese)

[59] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series **6**, Princeton University Press, Princeton, N. J., 1946.

[60] Z.-H. YANG, Y.-M. CHU, AND W. ZHANG, *Accurate approximations for the complete elliptic integral of the second kind*, J. Math. Anal. Appl. **438** (2016), no. 2, 875–888; available online at https://doi.org/10.1016/j.jmaa.2016.02.035.

[61] Z.-H. YANG AND J. TIAN, *Sharp inequalities for the generalized elliptic integrals of the first kind*, Ramanujan J. **48** (2019), no. 1, 91–116; available online at https://doi.org/10.1007/s11139-018-0061-4.

[62] Z.-H. YANG; J.-F. TIAN, AND Y.-R. ZHU, *A sharp lower bound for the complete elliptic integrals of the first kind*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 1, Paper No. 8, 17 pp.; available online at https://doi.org/10.1007/s13398-020-00949-6.

[63] Z.-H. YANG, M.-K. WANG, J.-F. TIAN, *Series expansion, higher-order monotonicity properties and inequalities for the modulus of the Grötzsch ring*, Proc. Edinburgh Math. Soc. Published online 2024:1-28; available online at https://doi.org/10.1017/S0013091524000440.

[64] L. YIN, X.-L. LIN, AND F. QI, *Monotonicity, convexity and inequalities related to complete (p,q,r)-elliptic integrals and generalized trigonometric functions*, Publ. Math. Debrecen **97** (2020), no. 1-2, 181–199; available online at https://doi.org/10.5486/PMD.2020.8793.

[65] L. YIN AND F. QI, *Some inequalities for complete elliptic integrals*, Appl. Math. E-Notes **14** (2014), 192–199.

[66] H.-C. ZHANG, B.-N. GUO, AND W.-S. DU, *On Qi's normalized remainder of Maclaurin power series expansion of logarithm of secant function*, Axioms **13** (2024), no. 12, Art. 860, 11 pages; available online at https://doi.org/10.3390/axioms13120860.

[67] G.-Z. ZHANG AND F. QI, *On convexity and power series expansion for logarithm of normalized tail of power series expansion for square of tangent*, J. Math. Inequal. **18** (2024), no. 3, 937–952; available online at https://doi.org/10.7153/jmi-2024-18-51.

[68] G.-Z. ZHANG, Z.-H. YANG, AND F. QI, *On normalized tails of series expansion of generating function of Bernoulli numbers*, Proc. Amer. Math. Soc. **153** (2025), no. 1, 131–141; available online at https://doi.org/10.1090/proc/16877.

[69] T. ZHANG, Z.-H. YANG, F. QI, AND W.-S. DU, *Some properties of normalized tails of Maclaurin power series expansions of sine and cosine*, Fractal Fract. **8** (2024), no. 5, Art. 257, 17 pages; available online at https://doi.org/10.3390/fractalfract8050257.

[70] X. ZHANG, G. WANG, AND Y. CHU, *Remarks on generalized elliptic integrals*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 2, 417–426; available online at https://doi.org/10.1017/S0308210507000327.

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