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COMPLEMENTARY AND REFINED INEQUALITIES FOR THE CAUCHY–SCHWARZ INEQUALITY INVOLVING MEANS

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Abstract. Let $(\mathscr{H},\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The well-known Cauchy-Schwarz inequality for the inner product asserts that $\left|\langle x,y\rangle\right| \leqslant \|x\|\|y\|$ for all $x,y\in\mathscr{H}$. In this paper, by using the consent of means, we obtain a refinement of the Cauchy-Schwarz inequality. Among other results, it is shown that, if $x,y\in\mathscr{H}$, $\mu,\nu\in[0,1]$, and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left|\langle x,y\rangle\right| \leqslant \frac{1}{p}\left|\langle x,y\rangle\right|^{1-\mu}\|x\|^{\mu}\|y\|^{\mu}+\frac{1}{q}\left|\langle x,y\rangle\right|^{\nu}\|x\|^{(1-\nu)}\|y\|^{1-\nu} \leqslant \|x\|\|y\|.$$

Moreover, we present a refinement of the classical Cauchy-Schwarz inequality. Furthermore, we obtain some numerical radius inequalities for the product of operators, which are interpolations of some earlier inequalities. For instance, if T is an operator on a Hilbert space $\mathscr{H}$, then we have

$$\begin{aligned} w^{2r}(T) &\leqslant \frac{1}{2^{\mu+1}p}w^{r(1-\mu)}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{\mu} \\ &\quad + \frac{1}{2^{2-\nu}p}w^{r\nu}(T^2)\left\||T|^{2r}+|T^*|^{2r}\right\|^{1-\nu}+\frac{1}{4}\left\||T|^{2r}+|T^*|^{2r}\right\| \\ &\leqslant \frac{1}{2}\left\||T|^{2r}+|T^*|^{2r}\right\| \end{aligned}$$

for $r\geqslant 1$, $\mu,\nu\in[0,1]$, and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$.

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