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ON REVERSE HARDY INEQUALITY IN VARIABLE LEBESGUE SPACES

ROVSHAN A. BANDALIYEV*, MEHRALY G. ALIYEV AND BAHRUZ K. AGARZAYEV

Abstract. In this paper, we prove the reverse Hardy inequality for Hardy operator in weighted variable Lebesgue spaces with exponent less than one. In particular, we establish necessary and sufficient conditions on weight functions for the validity of the reverse Hardy inequality for Hardy operator in weighted variable Lebesgue spaces with negative exponents. It should be noted that in the case of variable Lebesgue space $L_{p(x)}(0,\infty)$ for $0 < p(x) < 1$, the obtained necessary and sufficient conditions on the weight functions are different and coincide for some classes of variable exponents. Also, we prove similar results for the dual Hardy operator. The results are illustrated by an example.

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