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SOME NEW NUMERICAL RADIUS INEQUALITIES VIA AN IMPROVED VERSION OF KATO'S INEQUALITY

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Abstract. In this paper, we present new refinements of Kato's inequality. These refinements are then applied to derive improved upper bounds for specific numerical radius and norm inequalities. Our findings strengthen and extend several well-known numerical radius inequalities, providing more precise estimates than those previously established. In particular, we introduce two distinct refinement approaches based on improved versions of Young's inequality and a generalized Buzano's inequality. These results yield sharper bounds and wider applicability to bounded linear operators. Furthermore, we demonstrate the utility of these refinements through applications to triangle inequalities and Furuta-type inequalities, illustrating the robustness and versatility of our methods.

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