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INEQUALITIES INVOLVING A MEASURE OF BGM CLASS AND ZEROS OF CORRESPONDING ORTHOGONAL POLYNOMIALS

VIKASH KUMAR AND A. SWAMINATHAN

Abstract. Let $\tilde{\Phi}_n$ be a quasi-orthogonal polynomial of order 1 on the unit circle, obtained from an orthogonal polynomial Φ_n with measure μ, which belongs to BGM class, if there exists another measure $\tilde{\mu}$ such that $\tilde{\Phi}_n$ is a monic orthogonal polynomial. This article aims to investigate various properties related to BGM class. At first, we study the behaviour of the zeros of Φ_n and $\tilde{\Phi}_n$. Along with numerical examples, we analyze the zeros of Φ_n, corresponding para-orthogonal polynomial and its linear combination. Further, comparison of the norm inequalities among Φ_n and $\tilde{\Phi}_n$ are obtained by involving their measures. This leads to the study of the Lubinsky type inequality for the measures μ and $\tilde{\mu}$, without using the ordering relation between μ and $\tilde{\mu}$. Additionally, similar type of inequalities for the kernel type polynomials related to μ and $\tilde{\mu}$ are obtained.

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