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SHARP UPPER BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

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Abstract. The complete elliptic integral of the first kind, denoted by $\mathscr{K}(\cdot)$, is a class of special functions widely applied in mathematics, physics, and engineering. In this paper, we establish an upper bound for this function, given by

$$\mathscr{K}(r) \leqslant \frac{\pi}{2}(1-r^2)^{\frac{1}{2}(p_1+p_2r^2+p_3r^4+p_4r^6+p_5r^8)}$$

for all $r \in (0,1)$, where the parameters satisfy $p_1 \leqslant p_{1,0} = -1/2$, $p_2 \leqslant p_{2,0} = 1/32$, $p_3 \leqslant p_{3,0} = 1/64$, $p_4 \leqslant p_{4,0} = 251/24576$ and $p_5 \leqslant p_{5,0} = 123/16384$. Meanwhile, our results show that the parameters $p_{1,0}$, $p_{2,0}$, $p_{3,0}$, $p_{4,0}$ and $p_{5,0}$ are optimal and cannot be replaced by larger values. Finally, by utilizing the relationship between the complete elliptic integral of the first kind and the Gauss arithmetic-geometric mean, we establish sharp lower bounds for the Gauss arithmetic-geometric mean.

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