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LOGARITHMICALLY ABSOLUTE MONOTONICITY OF THE RATIO BETWEEN NORMALIZED REMAINDERS FOR A FUNCTION IN AN INTEGRAL REPRESENTATION OF THE RECIPROCAL OF THE GAMMA FUNCTION

YE SHUANG, CHUN-YING HE AND FENG QI*

Abstract. Let $\Phi(t)=1-\frac{t}{\tan t}+\ln\frac{t}{\sin t}$ for $t\in(-\pi,\pi)$. In the paper, in light of a theorem on signs of coefficients in power series and with aid of the Wronski formula, the authors prove that the normalized remainder $T_{2n+1}[\Phi(t)]$ for $n\in\mathbb{N}_0$ is a logarithmically absolutely monotonic function in $t\in(0,\pi)$ and a logarithmically completely monotonic function in $t\in(-\pi,0)$, that the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n\in\mathbb{N}_0$ is an absolutely monotonic function in $t\in(0,\pi)$ and a completely monotonic function in $t\in(-\pi,0)$, and that the normalized remainder $T_{2n+1}[\Phi(t)]$ and the ratio $\frac{T_{2n+3}[\Phi(t)]}{T_{2n+1}[\Phi(t)]}$ for $n\in\mathbb{N}_0$ can be extended analytically into the complex z-plane and are analytic in the disc $|z|<\pi$. Moreover, the authors expand $\frac{1}{\Phi(t)}$ for $0<|t|<\pi$ into a Laurent series. These results verify a guess and generalize the corresponding ones in a paper published on Math. Inequal. Appl. **28** (2025), no. 2, 343–354.

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