

NOVEL INEQUALITIES INVOLVING CONVEX FUNCTIONS AND THEIR APPLICATIONS

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Abstract. This paper presents a comprehensive generalization of the recent work by Yang and Zhang [J. Math. Inequal., **19** (2) (2025), 441–459]. We extend their piecewise interpolation approach originally developed for specific power-type functions to a broader setting involving general convex functions on the unit interval. By introducing new refinement and reverse inequalities, we establish sharper bounds for convex combinations and derive improved versions of classical results such as Jensen’s inequality and Young-type inequalities. Our methods incorporate convex analysis, interpolation theory, and weak submajorization techniques, leading to new applications in real and matrix analysis. In particular, we obtain refined inequalities for various matrix means, unitarily invariant norms, and numerical radius bounds, offering enhanced tools for use in operator theory, functional analysis, and quantum information theory.

Mathematics subject classification (2020): 47A12, 46C05, 47A05, 47A30.

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