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THE INTEGRAL ANALOGUE OF THE HARDY–LITTLEWOOD $L\log L$–INEQUALITY FOR BROWNIAN MOTION

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Abstract. Let $B = (B_t)_{t\geqslant 0}$ be standard Brownian motion started at zero. Then the following inequality is shown to be satisfied:

$$E\left(\max_{0\leqslant t\leqslant \tau} |B_t|\right) \leqslant c\,E\left(\int_0^\tau \frac{dt}{1+|B_t|}\right) + \frac{1}{2c-1}$$

for all stopping times τ for B and all $c > 1/2$. The stopping times at which the equality is attained are of the form:

$$\tau_c = \inf\left\{t>0 \mid S_t - \alpha X_t \geqslant \beta\right\}$$

where $\alpha = 1+1/(2c-1)$, $\beta = 1/(2c-1)$, $X_t = |B_t|$ and $S_t = \max_{0\leqslant r\leqslant t}|B_r|$. Taking infimum over all $c > 1/2$ we obtain:

$$E\left(\max_{0\leqslant t\leqslant \tau} |B_t|\right) \leqslant \frac{1}{2}E\left(\int_0^\tau \frac{dt}{1+|B_t|}\right) + \sqrt{2}\left(E\int_0^\tau \frac{dt}{1+|B_t|}\right)^{1/2}$$

for all stopping times τ for B. This inequality is sharp (the equality is attained at each τ_c for all $c > 1/2$). In view of Itô-Tanaka's formula these inequalities may be thought of as the integral analogues (for reflected Brownian motion) of the classical $L\log L$-inequality of Hardy and Littlewood. The proof is based upon solving the optimal stopping problem:

$$V = \sup_\tau E\left(S_\tau - cI_\tau\right)$$

where $I_\tau = \int_0^\tau (1+|B_t|)^{-1}\,dt$. The payoff V is shown to be finite if and only if $c > 1/2$, and in this case $V = 1/(2c-1)$. The optimal stopping problem is solved by applying the principle of smooth fit and the maximality principle. All results extend to the case when Brownian motion B starts at any given point.

Mathematics subject classification (1991): Primary 60G40, 60J65, 60E15. Secondary 60G44, 60J25, 60J60.

Key words and phrases: Brownian motion, integral of Brownian path, the $L\log L$-inequality of Hardy and Littlewood, optimal stopping (time), the principle of smooth fit, the maximality principle, Stephan's problem with moving boundary, Itô-Tanaka's formula, Burkholder-Gundy's inequality, Doob's maximal inequality, Doob's optional sampling theorem, local time.

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