THE INTEGRAL ANALOGUE OF THE HARDY–LITTLEWOOD $L \log L$–INEQUALITY FOR BROWNIAN MOTION

GORAN PESKIR

Abstract. Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion started at zero. Then the following inequality is shown to be satisfied:

$$E \left( \max_{0 \leq t \leq \tau} |B_t| \right) \leq c E \left( \int_0^\tau \frac{dt}{1 + |B_t|} \right) + \frac{1}{2c - 1}$$

for all stopping times $\tau$ for $B$ and all $c > 1/2$. The stopping times at which the equality is attained are of the form:

$$\tau_c = \inf \left\{ t > 0 \mid S_t - \alpha X_t \geq \beta \right\}$$

where $\alpha = 1 + 1/(2c - 1)$, $\beta = 1/(2c - 1)$, $X_t = |B_t|$ and $S_t = \max_{0 \leq r \leq t} |B_r|$. Taking infimum over all $c > 1/2$ we obtain:

$$E \left( \max_{0 \leq t \leq \tau} |B_t| \right) \leq \frac{1}{2} E \left( \int_0^\tau \frac{dt}{1 + |B_t|} \right) + \sqrt{2} \left( E \int_0^\tau \frac{dt}{1 + |B_t|} \right)^{1/2}$$

for all stopping times $\tau$ for $B$. This inequality is sharp (the equality is attained at each $\tau_c$ for all $c > 1/2$). In view of Itô-Tanaka’s formula these inequalities may be thought of as the integral analogues (for reflected Brownian motion) of the classical $L \log L$-inequality of Hardy and Littlewood. The proof is based upon solving the optimal stopping problem:

$$V = \sup \tau E \left( S_\tau - cI_\tau \right)$$

where $I_\tau = \int_0^\tau (1 + |B_t|)^{-1} dt$. The payoff $V$ is shown to be finite if and only if $c > 1/2$, and in this case $V = 1/(2c - 1)$. The optimal stopping problem is solved by applying the principle of smooth fit and the maximality principle. All results extend to the case when Brownian motion $B$ starts at any given point.


Key words and phrases: Brownian motion, integral of Brownian path, the $L \log L$-inequality of Hardy and Littlewood, optimal stopping (time), the principle of smooth fit, the maximality principle, Stephan’s problem with moving boundary, Itô-Tanaka’s formula, Burkholder-Gundy’s inequality, Doob’s maximal inequality, Doob’s optional sampling theorem, local time.

REFERENCES


