

## THE $s$ -CONVEX ORDERS AMONG REAL RANDOM VARIABLES, WITH APPLICATIONS

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*Abstract.* In this paper, new classes of stochastic order relations are introduced. These can be seen as extensions of the usual convex order and are closely related to the orderings discussed in Lefèvre and Utev (1996), as well as to the stochastic dominances in economics and stop-loss orders in actuarial sciences. These classes are studied in detail, including properties, characterizations, sufficient conditions, and extrema with respect to these orderings in different sets of distribution functions. Some applications illustrate the theory.

*Mathematics subject classification* (1991): 60E15.

*Key words and phrases:*  $s$ -convex orders,  $s$ -increasing convex orders, convexity of higher order, Tchebycheff systems, queueing theory, insurance.

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