GENERALIZED FURUTA INEQUALITY IN
BANACH *-ALGEBRAS AND ITS APPLICATIONS

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Abstract. Okayasu [12] proved the useful Löwner-Heinz inequality in Banach *-algebra as follows. Let $A$ be a unital hermitian Banach *-algebra with continuous involution and $a, b \in A$. If $a \geq b > 0$, then $a^p \geq b^p$ for $p \in (0, 1]$. For $a > 0$, $a^\alpha = \exp(\alpha \log a)$, where log is the principal branch of the complex logarithm. As a nice application of this result, K.Tanahashi and M.Uchiyama [15] proved the following very interesting inequality. Let $a, b \in A$. Let $R \ni p, q, r \geq 0$ satisfy $(1 + r)q \geq p + r$ and $q \geq 1$.

$$\left( b^\frac{r}{q} a^p b^\frac{r}{q} \right)^\frac{1}{s} \geq \left( b^\frac{r}{q} b^p b^\frac{r}{q} \right)^\frac{1}{s} \quad \text{if } a \geq b > 0.$$ 

This inequality may be called to be “Banach *-algebra version” of Furuta inequality. By using this result and Löwner-Heinz inequality in Banach *-algebra in Okayasu [12], we show the following generalized Furuta inequality. Let $a, b \in A$. If $a \geq b > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$

$$a^q t^{-r} \geq \left( a^{\frac{q}{p}} (a^{\frac{p}{q}} b^p a^{\frac{-p}{q}}) a^{-1} \right)^{\frac{q-1}{q}} a^{-\frac{r}{t}}$$

holds for $s \geq 1$ and $r \geq t$. Moreover as an application of this inequality, we show that if $a \geq b > 0$, for each $t \in [0, 1]$, $q \geq 0$ and $p \geq t$,

$$G_{p,q,t}(a, b, r, s) = a^{-s} \left( a^{\frac{q}{p}} (a^{\frac{p}{q}} b^p a^{\frac{-p}{q}}) a^{-1} \right)^{\frac{q-1}{q}} a^{-\frac{r}{t}}$$

is decreasing for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q - t$.


Key words and phrases: Löwner-Heinz inequality, Furuta inequality, Banach *-algebra.

REFERENCES

[4] T. FURUTA, A \geq B \geq 0 assures $\left( B^p A^p B^q \right)^{1/q} \geq B^{(p+2q)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.


