Mathematical
Inequalities
& Applications
Volume 2, Number 2 (1999), 289–295

GENERALIZED FURUTA INEQUALITY IN BANACH $*$–ALGEBRAS AND ITS APPLICATIONS

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Abstract. Okayasu [12] proved the useful Löwner-Heinz inequality in Banach $*$-algebra as follows. Let A be a unital hermitian Banach $*$-algebra with continuous involution and $a, b \in A$. If $a \geqslant b > 0$, then $a^p \geqslant b^p$ for $p \in (0,1]$. For $a > 0$, a^{α} =exp($\alpha \log a$),where log is the principal branch of the complex logarithm. As a nice application of this result, K.Tanahashi and M.Uchiyama [15] proved the following very interesting inequality. Let $a, b \in A$. Let $R \ni p, q, r \geqslant 0$ satisfy $(1+r)q \geqslant p+r$ and $q \geqslant 1$.

$$(b^{\frac{r}{2}} a^p b^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (b^{\frac{r}{2}} b^p b^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{if } a \geqslant b > 0.$$

This inequality may be called to be "Banach $*$-algebra version" of Furuta inequality. By using this result and Löwner-Heinz inequality in Banach $*$-algebra in Okayasu [12], we show the following generalized Furuta inequality. Let $a, b \in A$. If $a \geqslant b > 0$, then for each $1 \geqslant q \geqslant t \geqslant 0$ and $p \geqslant q$

$$a^{q-t+r} \geqslant \{a^{\frac{r}{2}} (a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for $s \geqslant 1$ and $r \geqslant t$. Moreover as an application of this inequality, we show that if $a \geqslant b > 0$, for each $t \in [0,1]$, $q \geqslant 0$ and $p \geqslant t$,

$$G_{p,q,t}(a,b,r,s) = a^{\frac{-r}{2}} \{a^{\frac{r}{2}} (a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} a^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$ such that $(p-t)s \geqslant q-t$.

Mathematics subject classification (1991): 47A63, 47A05, 47C10.
Key words and phrases: Löwner-Heinz inequality, Furuta inequality, Banach $*$-algebra.

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Paper MIA-02-26