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ON MIXED HÖLDER–MINKOWSKI INEQUALITIES AND TOTAL CONVEXITY OF CERTAIN FUNCTIONS IN $\mathscr{L}^p(\Omega)$

C. A. ISNARD AND A. N. IUSEM *

Abstract. We prove the following mixed Hölder-Minkowski-type inequalities for all $x,z \in \mathscr{L}^p(\Omega)$:

$$0 \leqslant \frac{||z||_p^{p-s}}{s}\Big[\Big(||x||_p + ||z||_p\Big)^s - ||x+z||_p^s\Big] \leqslant ||x||_p||y||_q - \mathrm{Re}(\langle x,y\rangle) \quad \text{if} \;\; 1 \leqslant s \leqslant p \leqslant 2,$$

$$\frac{||z||_p^{p-s}}{s}\Big[\Big(||x||_p + ||z||_p\Big)^s - ||x+z||_p^s\Big] \geqslant ||x||_p||y||_q - \mathrm{Re}(\langle x,y\rangle) \geqslant 0 \;\; \text{if} \;\; 2 \leqslant p \leqslant s,$$

where $y \in \mathscr{L}^q(\Omega)$ is defined as $y(\xi) = |z(\xi)|^{p-2}z(\xi)$, if $z(\xi) \neq 0$, $y(\xi) = 0$ otherwise, and $1/p + 1/q = 1$. Next we consider the Bregman distance $D_f : \mathscr{L}^p(\Omega) \times \mathscr{L}^p(\Omega) \to \mathbf{R}$ defined as $D_f(x,y) = f(x) - f(y) - \langle f'(y), x-y\rangle$, with $f(x) = ||x||_p^s$ ($s,p > 1$), and prove that $\inf\{D_f(u,z) : ||u-z||_p = t\} > 0$, $\sup\{D_f(u,z) : ||u-z||_p = t\} < \infty$, for all $p,s > 1$, all $z \in \mathscr{L}^p(\Omega)$ and all $t > 0$, so that the Bregman distance induced by $f(x) = ||x||_p^s$ and the metric distance $d(x,y) = ||x-y||_p$ are topologically equivalent. As a consequence, this f can be used in projection algorithms for the convex feasibility problem and generalized proximal point methods for convex optimization in $\mathscr{L}^p(\Omega)$.

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