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BEST POSSIBLE BOUNDS FOR ORDERED POSITIVE NUMBERS USING THEIR SUM AND PRODUCT

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Abstract. Best possible bounds for real numbers $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$ with prescribed sum $a = \lambda_1 + \cdots + \lambda_n$ and product $d = \lambda_1 \cdots \lambda_n$ are presented. These bounds can be expressed algebraically only in certain special cases. In the general case, explicit bounds are found by using extra bounds. The results are applied to eigenvalue estimation, when the λ_k's are regarded as eigenvalues of an n by n matrix A, $a = \operatorname{tr} A$, and $d = \det A$. The case when the eigenvalues are real but not necessarily positive is also discussed. The bounds are compared with bounds using a and $b = \lambda_1^2 + \cdots + \lambda_n^2$; i.e, with eigenvalue bounds using $\operatorname{tr} A$ and $\operatorname{tr} A^2$.

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