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ON EMBEDDINGS BETWEEN CLASSICAL LORENTZ SPACES

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Abstract. Let $p \in (0,\infty)$, let v be a weight on $(0,\infty)$ and let $\Lambda^p(v)$ be the classical Lorentz space, determined by the norm $\|f\|_{\Lambda^p(v)} := (\int_0^\infty (f^*(t))^p v(t)\,dt)^{1/p}$. When $p \in (1,\infty)$, this space is known to be a Banach space if and only if v is non-increasing, while it is only equivalent to a Banach space if and only if $\Lambda^p(v) = \Gamma^p(v)$, where $\|f\|_{\Gamma^p(v)} := (\int_0^\infty (f^{**}(t))^p v(t)\,dt)^{1/p}$. We may thus conclude that, for $p \in (1,\infty)$, the space $\Lambda^p(v)$ is equivalent to a Banach space if and only if the norm of a function f in it can be expressed in terms of f^{**}. We study the question whether an analogous assertion holds when $p = 1$. Motivated by this problem, we consider general embeddings between four types of classical and weak Lorentz spaces, namely, $\Lambda^p(v)$, $\Lambda^{p,\infty}(v)$, $\Gamma^p(v)$, $\Gamma^{p,\infty}(v)$, where $\Lambda^{p,\infty}(v)$ and $\Gamma^{p,\infty}(v)$ are certain weak analogues of the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$, respectively. We present a unified approach to these embeddings, based on rearrangement techniques. We survey all the known results and prove new ones. Our main results concern the embedding $\Gamma^{p,\infty}(v) \hookrightarrow \Lambda^q(w)$ which had not been characterized so far. We apply our results to the characterization of associate spaces of classical and weak Lorentz spaces and we give a characterization of fundamental functions for which the endpoint Lorentz space and the endpoint Marcinkiewicz space coincide.

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