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MONOTONICITY RESULTS AND INEQUALITIES FOR THE GAMMA AND INCOMPLETE GAMMA FUNCTIONS

FENG QI

Abstract. In the article, using the monotonicity and inequalities of the generalized weighted mean values with two parameters, we prove that the functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ are increasing in $r>0$, $s>0$ and $x>0$, where $\Gamma(s)$, $\Gamma(s,x)$ and $\gamma(s,x)$ denote the gamma and incomplete gamma functions with usual notation. From this, some monotonicity results and inequalities for the gamma or incomplete gamma functions are deduced or extended, a unified proof of some known results for the gamma function is given.

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