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SOME INEQUALITIES AND EMBEDDINGS FOR WEIGHTED W_0 SPACES ON DOMAINS WITH FRACTAL BOUNDARIES

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Abstract. If Ω is a finite measure domain we show that several Poincaré, Hardy-type, or multiplicative inequalities as well as classical Sobolev embedding theorems on $W_0^{m,p}(\Omega)$ may be extended to versions with singular or degenerate weights involving powers of the distance to the boundary function provided that $\partial\Omega$ is "fractal" in the sense that $\partial\Omega$ has interior Minkowski dimension $\widetilde{M}_D(\partial\Omega) < n$. For unbounded non-finite measure domains such extensions may also often be made if $\partial\Omega$ satisfies a certain definition of "locally fractal".

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