AN EXTENSION OF UCHIYAMA’S RESULT ASSOCIATED WITH AN ORDER PRESERVING OPERATOR INEQUALITY

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Abstract. Let $A$, $B$ and $C$ be positive invertible operators and also let $r, s$ and $t$ be non-negative real numbers such that $t \geq s$ and $(r,t) \neq (0,0)$. Then the following (I) and (II) hold and follows from each other.

(I) If $A^t \leq B^s \nabla^t \lambda C^t$ (i.e., $\log A^t \leq \log (B^s \nabla^t \lambda C^t)$) for all $t \geq 0$, then

$$f(t) = \{A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}}\}^{\frac{r}{r+s}}$$

is an increasing function of $t$.

(II) If $A^t \geq B^s \nabla^t \lambda C^t$ (i.e., $\log A^t \geq \log (B^s \nabla^t \lambda C^t)$) for all $t \geq 0$, then

$$h(t) = \{A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}}\}^{\frac{r}{r+s}}$$

is a decreasing function of $t$,

where $B^s \nabla^t \lambda C$ and $B^s \nabla^t \lambda C$ are the arithmetic mean and the harmonic mean respectively.

In particular we have

(1') If $A^t \leq B^s \nabla^t \lambda C^t$, then

$$A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}} \leq \{A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}}\}^{\frac{r}{r+s}}$$

(II') If $A^t \geq B^s \nabla^t \lambda C^t$, then

$$A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}} \geq \{A^{\frac{t}{r}} (B^s \nabla^t \lambda C^t)A^{\frac{t}{s}}\}^{\frac{r}{r+s}}.$$

These are extensions of the recent results in Uchiyama [12].


Key words and phrases: Chaotic order, Furuta inequality, arithmetic mean and harmonic mean.

REFERENCES

[5] T. Furuta, $A \geq B \geq 0$ assures $(B^p \nabla^q B^p)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.