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INEQUALITITES FOR CAUCHY MEAN VALUES

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Abstract. The Cauchy Mean Value Theorem for divided differences (see e.g. [16]) states the following:

Suppose that $x_1 \leqslant \cdots \leqslant x_n$ *and* $f^{(n-1)}, g^{(n-1)}$ *exist, with* $g^{(n-1)} \neq 0$, *on* $[x_1, x_n]$. *Then there is a* $t \in [x_1, x_n]$ *(moreover* $t \in (x_1, x_n)$ *if* $x_1 < x_n$ *) such that*

$$\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g} = \frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}$$

where $[x_1, \dots, x_n]_f$ *denotes the divided difference of* f *at the points* $x_1, \dots, x_n$.

If the function $\dfrac{f^{(n-1)}}{g^{(n-1)}}$ is invertible then

$$t = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1} \left(\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g}\right)$$

is a mean value of $x_1, \dots, x_n$. It is called the *Cauchy mean of the numbers* $x_1, \dots, x_n$ and will be denoted by $\mathcal{D}_{f,g}(x_1, \dots, x_n)$.

In this survey paper we discuss the equality, homogeneity of Cauchy means and inequalities of general nature: comparison, Minkowski's inequality of (homogeneous) Stolarsky's means and also the comparison and general comparison of Cauchy means.

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