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CHARACTERIZATIONS OF CHAOTIC ORDER ASSOCIATED WITH THE MOND–SHISHA DIFFERENCE

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Abstract. Recently, Yamazaki showed new order preserving operator inequalities on the usual order and the chaotic order by estimating the lower bound of the difference. Mond and Shisha gave an estimtate of the difference of the arithmetic one to the geometric one, as a converse of the arithmetic-geometric mean inequality. In this paper, by means of the Mond-Shisha difference, we shall put another interpretation on a characterization of the chaotic order associated with the difference by Yamazaki: If $A > 0$, $MI \geqslant B \geqslant mI > 0$ and $h = \frac{M}{m} > 1$, then $\log A \geqslant \log B$ is equivalent to

$$A^p + D(m^p, M^p)I \geqslant B^p \qquad \text{for all } p > 0,$$

where

$$D(m^p, M^p) = \theta M^p + (1-\theta)m^p - M^{p\theta} m^{p(1-\theta)} \quad \text{and} \quad \theta = \log\left(\frac{h^p - 1}{p\log h}\right)\frac{1}{p\log h}.$$

Moreover, inspired by Yamazaki's work, we shall make an attempt to clarify distinction between the usual order and the chaotic order by using the Furuta inequality. Among others, we show the following parametrized order preserving operator inequalities associated with the difference: If $A > 0$ and $MI \geqslant B \geqslant mI > 0$, then for each $\delta \in [0, 1]$

$$A^\delta \geqslant B^\delta \quad \text{if and only if} \quad A^{p+\delta} + \frac{1}{m^r} C(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta})I \geqslant B^{p+\delta} \ \text{ for } p, r > 0$$

where the case $\delta = 0$ means the chaotic order.

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