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# CONVEXITY OR CONCAVITY INEQUALITIES FOR HERMITIAN OPERATORS

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*Abstract.* Given a Hermitian operator, a monotone convex function $f$ and a subspace $\mathscr{E}$, $\dim \mathscr{E} < \infty$, there exists a unitary operator $U$ on $\mathscr{E}$ such that $f(A_{\mathscr{E}}) \leqslant Uf(A)_{\mathscr{E}}U^*$. (Here $X_{\mathscr{E}}$ denotes the compression of $X$ onto $\mathscr{E}$). A related result is: For a monotone convex function $f$, $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, and Hermitian operators $A$, $B$ on a finite dimensional space, there exists a unitary $U$ such that $f(\alpha A + \beta B) \leqslant U\{\alpha f(A) + \beta f(B)\}U^*$. More general convexity results are established. Also, several old and new trace inequalities of Brown-Kosaki and Hansen-Pedersen type are derived. We study the behaviour of the map $p \longrightarrow \{(A^p)_{\mathscr{E}}\}^{1/p}$, $A \geqslant 0$, $0 < p < \infty$.

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