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HOW DEVIANT CAN YOU BE? THE COMPLETE SOLUTION

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Abstract. We consider the problem of optimal deterministic lower and upper bounds on arbitrary linear combinations of order statistics centered about the sample mean in units generated by the sample central absolute moments of various orders. The signs of the evaluations depend merely on the coefficients of the linear combinations. Hitherto all the positive upper and negative lower bounds have been established as well as a few exceptional positive lower and negative upper ones. In the paper, we complete the solution by presenting all the positive lower bounds and negative upper bounds and respective samples attaining them. We also specify the general results by considering several important examples.

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