Mathematical
Inequalities
& Applications
Volume 10, Number 4 (2007), 761–769

CIRCULAR INTERLACING WITH RECIPROCAL POLYNOMIALS

PIROSKA LAKATOS AND LÁSZLÓ LOSONCZI

Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geqslant 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, if there is a $B \in \mathbb{R}$ such that $A_m B \geqslant 0, |A_m| \geqslant |B|$ and

$$|A_m + B| \geqslant \sum_{k=1}^{m-1} |A_k + B - A_m|$$

holds.

If the inequality is strict then the zeros of P_m have the form $e^{\pm u_j}$ $(j = 1, \dots, \left[\frac{m}{2}\right])$ where

$$\frac{2(j-1)\pi}{m} < u_j < \frac{2j\pi}{m} \quad (j = 1, \dots, \left[\frac{m}{2}\right])$$

and they are simple (for odd m, in addition to these zeros, $-1 = e^{-i\pi}$ is a zero too).

This implies that the polynomial P_m (with $A_m > 0$) and $z^{2m} - 1$ satisfy the circular interlacing condition.

If in the inequality (for the coefficients) equality holds, then double zeros may arise, we discuss how this can happen.

Mathematics subject classification (2000): 30C15, 12D10, 42C05.
Key words and phrases: reciprocal polynomials, zeros on the unit circle, circular interlacing.

REFERENCES

[1] E. M. BONSALL, M. MARDEN, *Zeros of self-inversive polynomials*, Proc. Amer. Math. Soc., **3**, (1952), 471–475.
[2] A. COHN, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Zeit., **14**, (1922), 110–148.
[3] P. LAKATOS, *On zeros of reciprocal polynomials*, Publ. Math. Debrecen, **61**, (2002), 645–661.
[4] P. LAKATOS, L. LOSONCZI, *Self-inversive polynomials whose zeros are on the unit circle*, Publ. Math. Debrecen, **65**, (2004), 409–420.
[5] M. MARDEN, *Geometry of polynomials*, Math. Surveys No. 3, Amer. Math. Soc. Providence, Rhode Island 1966.
[6] J. MCKEE, C. SMYTH, *There are Salem numbers of every trace*, Bull. London Math. Soc., **37**, (2005), 25–36.
[7] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ, AND TH. M. RASSIAS, *Topics in polynomials*, World Scientific, Singapore-New Jersey-London-Hong Kong, 1994.
[8] T. J. RIVLIN, *Chebyshev polynomials*, A Wiley-Interscience Publication, 1990.
[9] A. SCHINZEL, *Self-inversive polynomials with all zeros on the unit circle*, Ramanujan J., **9**, (2005), 19–23.