

SOME INEQUALITIES INVOLVING GENERALIZED BESSEL FUNCTIONS

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Abstract. Let u_p be the generalized and normalized Bessel function depending on parameters b, c, p and let $\lambda_p(x) = u_p(x^2)$, $x \in \mathbb{R}$. In this paper we extend to the function λ_p some well-known classical inequalities like Mahajan's inequality, Mitrinović's inequality, improvements of Jordan's inequality, Redheffer's inequality, using an adequate integral representation of the function λ_p and the monotone form of l'Hospital's rule. Moreover we prove that the integral

$$\zeta_p(x) = \int_0^x \lambda_p(t) dt$$

is sub-additive (super-additive) under certain conditions on parameters b, c, p .

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