

REFINEMENTS, EXTENSIONS AND GENERALIZATIONS OF THE SECOND KERSHAW'S DOUBLE INEQUALITY

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Abstract. In the paper, the second Kershaw's double inequality concerning the ratio of two gamma functions is refined, extended and generalized elegantly.

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