

## A CONVERSE OF THE HÖLDER INEQUALITY THEOREM

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*Abstract.* Let  $(\Omega, \Sigma, \mu)$  be a measure space such that  $0 < \mu(A) < 1 < \mu(B) < \infty$  for some  $A, B \in \Sigma$  and let bijections  $\varphi_1, \varphi_2, \psi_1, \psi_2 : (0, \infty) \rightarrow (0, \infty)$  be such that  $\frac{\psi_1 \circ \varphi_1(t)}{t} \leq c \leq \frac{t}{\psi_2 \circ \varphi_2(t)}$  ( $t > 0$ ). We prove that if

$$\int_{\Omega} xy d\mu \leq \psi_1 \left( \int_{\Omega(\mathbf{x})} \varphi_1 \circ |x| d\mu \right) \psi_2 \left( \int_{\Omega(\mathbf{y})} \varphi_2 \circ |y| d\mu \right)$$

for all nonnegative  $\mu$ -integrable simple functions  $\mathbf{x}, \mathbf{y} : \Omega \rightarrow \mathbb{R}$  (where  $\Omega(\mathbf{x})$  stands for the support of  $\mathbf{x}$ ), then there exists a real  $p > 1$  such that

$$\frac{\varphi_1(t)}{\varphi_1(1)} = t^p, \quad \frac{\psi_1(t)}{\psi_1(1)} = t^{1/p}, \quad \frac{\varphi_2(t)}{\varphi_2(1)} = t^q, \quad \frac{\psi_2(t)}{\psi_2(1)} = t^{1/q}, \quad t > 0,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . A relevant result for the reversed inequality is also given.

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