

# HERMITE–HADAMARD’S INEQUALITY AND THE $p$ –HH–NORM ON THE CARTESIAN PRODUCT OF TWO COPIES OF A NORMED SPACE

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**Abstract.** The Cartesian product of two copies of a normed space is naturally equipped with the well-known  $p$ -norm. In this paper, another notion of norm is introduced, and will be called the  $p$ -HH-norm. This norm is an extension of the generalised logarithmic mean and is connected to the  $p$ -norm by the Hermite-Hadamard’s inequality. The Cartesian product space (with respect to both norms) is complete, when the (original) normed space is. A proof for the completeness of the  $p$ -HH-norm via Ostrowski’s inequality is provided. This space is embedded as a subspace of the well-known Lebesgue-Bochner function space (as a closed subspace, when the norm is a Banach norm). Consequently, its geometrical properties are inherited from those of Lebesgue-Bochner space. An explicit expression of the superior (inferior) semi-inner product associated to both norms is considered and used to provide alternative proofs for the smoothness and reflexivity of this space.

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