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SOBOLEV EMBEDDINGS FOR UNBOUNDED DOMAIN WITH VARIABLE EXPONENT HAVING VALUES ACROSS N

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Abstract. We study Sobolev embeddings for unbounded domain with variable exponent having values across N. A main result of this paper is the following theorem: Let Ω be an unbounded domain in $\mathbb{R}^N$ satisfying the uniform cone condition. Suppose that $p:\overline{\Omega}\to\mathbb{R}$ is Lipschitz and $1<\inf_\Omega p\leqslant\sup_\Omega p<\infty$. Then there holds a continuous embedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ for any $q\in L^\infty(\Omega)$ satisfying condition $p(x)\leqslant q(x)\leqslant p^*(x)$ for a.e. $x\in\Omega$, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$ and $p^*(x)=\infty$ if $p(x)\geqslant N$. In this theorem the usual condition $\sup p(x)<N$ is not required.

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