

## REAL AND COMPLEX OPERATOR NORMS BETWEEN QUASI-BANACH $L^p - L^q$ SPACES

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*Abstract.* Relations between the norms of an operator and its complexification as a mapping from  $L^p$  to  $L^q$  has been recognized as a serious problem in analysis after the publication of Marcel Riesz's work on convexity and bilinear forms in 1926. We summarize here what it is known about these relations in the case of normed Lebesgue spaces and investigate the quasi-normed case, i. e. we consider all  $0 < p, q \leq \infty$ . In particular, in the lower triangle, that is, for  $0 < p \leq q \leq \infty$  these norms are the same. In the upper triangle and the normed case, that is, when  $1 \leq q < p \leq \infty$  the norm of the complexification of a real operator is obviously not bigger than 2 times its real norm. In 1977 Krivine proved that the constant 2 can be replaced by  $\sqrt{2}$ . On the other hand, it was suspected that in the case of quasi-normed Lebesgue spaces ( $0 < q < p \leq \infty$ ) the corresponding constant could be arbitrarily large, but as we will see this is not the case. More precisely, we prove that this constant for quasi-normed Lebesgue spaces is between 1 and 2. Some additional properties and estimates of this constant with some results about the relation between complex and real norms of operators, including those between two-dimensional Orlicz spaces are presented in the first four chapters. Finally, in Chapter 5, we use the results on the estimates of the norms in the proof of the real Riesz-Thorin interpolation theorem valid in the first quadrant.

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