

SOME BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

BAI-NI GUO AND FENG QI

Abstract. In the article, the complete elliptic integrals of the first and second kinds are bounded by using the power series expansions of some functions, the celebrated Wallis inequality, and an integral inequality due to R. P. Agarwal, P. Cerone, S. S. Dragomir and F. Qi.

Mathematics subject classification (2010): 26D15, 33C75, 33E05.

Keywords and phrases: Bounds, inequality, complete elliptic integral, the first kind, the second kind, Wallis formula, integral inequality.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN (EDS.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Reprint of the 1972 edition. A Wiley-Interscience Publication. Selected Government Publications. John Wiley & Sons, Inc., New York; National Bureau of Standards, Washington, DC, 1984.
- [2] R. P. AGARWAL AND S. S. DRAGOMIR, *An application of Hayashi's inequality for differentiable functions*, *Computers Math. Appl.*, **32**, 6 (1996), 95–99.
- [3] G. ALMKVIST AND B. BERNDT, *Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π , and the Ladies Diary*, *Amer. Math. Monthly*, **95** (1988), 585–607.
- [4] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, *SIAM J. Math. Anal.*, **23** (1992), 512–524.
- [5] Á. BARICZ, *Turán type inequalities for generalized complete elliptic integrals*, *Math. Z.*, **256**, 4 (2007), 895–911.
- [6] Á. BARICZ, *Turán type inequalities for hypergeometric functions*, *Proc. Amer. Math. Soc.*, **136**, 9 (2008), 3223–3229.
- [7] J. CAO, D.-W. NIU, AND F. QI, *A Wallis type inequality and a double inequality for probability integral*, *Austral. J. Math. Anal. Appl.*, **5**, 1 (2007), Art. 3; Available online at URL: <http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/v4I1P3.tex>.
- [8] P. CERONE AND S. S. DRAGOMIR, *Lobatto type quadrature rules for functions with bounded derivative*, *Math. Inequal. Appl.*, **3**, 2 (2000), 197–209.
- [9] CH.-P. CHEN, *On Wallis' inequality*, *Commun. Math. Anal.*, **1**, 1 (2006), 1–5.
- [10] CH.-P. CHEN, *Proof of the best bounds in Wallis' inequality*, *Gen. Math.*, **13**, 2 (2005), 117–120.
- [11] CH.-P. CHEN AND F. QI, *Best upper and lower bounds in Wallis' inequality*, *J. Indones. Math. Soc. (MIHMI)*, **11**, 2 (2005), 137–141.
- [12] CH.-P. CHEN AND F. QI, *Completely monotonic function associated with the gamma function and proof of Wallis' inequality*, *Tamkang J. Math.*, **36**, 4 (2005), 303–307.
- [13] CH.-P. CHEN AND F. QI, *The best bounds in Wallis' inequality*, *Proc. Amer. Math. Soc.*, **133**, 2 (2005), 397–401.
- [14] CH.-P. CHEN AND F. QI, *The best bounds to $\frac{(2n)!}{2^{2n}(n!)^2}$* , *Math. Gaz.*, **88** (2004), 54–55.
- [15] R. H. GARSTANG, *Bounds on an elliptic integral*, *Amer. Math. Monthly*, **94**, 6 (1987), 556–557.
- [16] B.-N. GUO AND F. QI, *Estimates for an integral in L^p norm of the $(n+1)$ -th derivative of its integrand*, *RGMIA Res. Rep. Coll.*, **3**, 3 (2000), Art. 2, 359–363; Available online at URL: <http://rgmia.org/v3n3.php>.

- [17] B.-N. GUO AND F. QI, *Estimates for an integral in L^p norm of the $(n+1)$ -th derivative of its integrand*, Inequality Theory and Applications, Volume 3, 127–131, Nova Science Publishers, Hauppauge, NY, 2003.
- [18] B.-N. GUO AND F. QI, *Some bounds for the complete elliptic integrals of the first and second kinds*, Available online at URL: <http://arxiv.org/abs/0905.2787>.
- [19] B.-N. GUO AND F. QI, *Some estimates of an integral in terms of the L^p -norm of the $(n+1)$ st derivative of its integrand*, Anal. Math., **29**, 1 (2003), 1–6.
- [20] B.-N. GUO, F. QI AND SH.-J. JING, *Improvement for the upper bound of a class of elliptic integral*, Jiàozuò Kuàngyè Xuéyuàn Xuébào (Journal of Jiaozuo Mining Institute), **14**, 6 (1995), 125–128. (Chinese)
- [21] D. K. KAZARINOFF, *On Wallis' formula*, Proc. Edinburgh Math. Soc., **9** (1956), Edinburgh Math. Notes No. 40 (1956), 19–21.
- [22] S. KOUMANDOS, *Remarks on a paper by Chao-Ping Chen and Feng Qi*, Proc. Amer. Math. Soc., **134** (2006), 1365–1367.
- [23] J.-CH. KUANG, *Chángyòng Bùděngshì (Applied Inequalities)*, 2nd ed., Húnán Jiàoyù Chūbǎn Shè (Hunan Education Press), Changsha, China, May 1993. (Chinese)
- [24] J.-CH. KUANG, *Chángyòng Bùděngshì (Applied Inequalities)*, 3rd ed., Shandong Science and Technology Press, Jí'nán City, Shandong Province, China, 2004. (Chinese)
- [25] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
- [26] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl., **2010** (2010), Article ID 493058, 84 pages; Available online at URL: <http://dx.doi.org/10.1155/2010/493058>.
- [27] F. QI, *Bounds for the ratio of two gamma functions – From Wendel's limit to Elezović-Giordano-Pečarić's theorem*, Available online at URL: <http://arxiv.org/abs/0902.2514>.
- [28] F. QI, *Bounds for the ratio of two gamma functions – From Wendel's and related inequalities to logarithmically completely monotonic functions*, Available online at URL: <http://arxiv.org/abs/0904.1048>.
- [29] F. QI, *Further generalizations of inequalities for an integral*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **8** (1997), 79–83.
- [30] F. QI, *Inequalities for a multiple integral*, Acta Math. Hungar., **84** (1999), no. 1-2, 19–26.
- [31] F. QI, *Inequalities for a weighted multiple integral*, RGMIA Res. Rep. Coll., **2**, 7 (1999), Art. 4, 991–997; Available online at URL: <http://rgmia.org/v2n7.php>.
- [32] F. QI, *Inequalities for a weighted multiple integral*, J. Math. Anal. Appl., **253**, 2 (2001), 381–388.
- [33] F. QI, *Inequalities for an integral*, Math. Gaz., **80**, 488 (1996), 376–377.
- [34] F. QI, P. CERONE, AND S. S. DRAGOMIR, *Some new Iyengar type inequalities*, Rocky Mountain J. Math., **35**, 3 (2005), 997–1015.
- [35] F. QI, L.-H. CUI, AND S.-L. XU, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl., **2**, 4 (1999), 517–528.
- [36] F. QI AND B.-N. GUO, *Estimate for upper bound of an elliptic integral*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory), **26**, 3 (1996), 285–288. (Chinese)
- [37] F. QI AND B.-N. GUO, *The estimation of inferior bound for an ellipse integral*, Gōngkē Shùxué (Journal of Mathematics for Technology), **10**, 1 (1994), 87–90. (Chinese)
- [38] F. QI AND ZH. HUANG, *Inequalities of the complete elliptic integrals*, Tamkang J. Math., **29**, 3 (1998), 165–169.
- [39] F. QI, D.-W. NIU AND B.-N. GUO, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl., **2009** (2009), Article ID 271923, 52 pages; Available online at URL: <http://dx.doi.org/10.1155/2009/271923>.
- [40] F. QI AND A. SOFO, *An alternative and united proof of a double inequality for bounding the arithmetic-geometric mean*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **71**, 3 (2009), 69–76.
- [41] F. QI AND Y.-J. ZHANG, *Inequalities for a weighted integral*, RGMIA Res. Rep. Coll., **2**, 7 (1999), Art. 2, 967–975; Available online at URL: <http://rgmia.org/v2n7.php>.
- [42] F. QI AND Y.-J. ZHANG, *Inequalities for a weighted integral*, Adv. Stud. Contemp. Math. (Kyungshang), **4**, 2 (2002), 93–101.
- [43] TH. M. RASSIAS, *Problem E 3111*, Amer. Math. Monthly, **92**, 9 (1985), 665.
- [44] J.-SH. SUN AND CH.-M. QU, *Alternative proof of the best bounds of Wallis' inequality*, Commun. Math. Anal., **2**, 1 (2007), 23–27.

- [45] G. N. WATSON, *A note on gamma functions*, Proc. Edinburgh Math. Soc., **11**, 2 (1958/1959), Edinburgh Math Notes No. 42 (misprinted 41) (1959), 7–9.
- [46] E. W. WEISSTEIN, *Wallis Cosine Formula*, From MathWorld – A Wolfram Web Resource; Available online at URL: <http://mathworld.wolfram.com/WallisFormula.html>.
- [47] L.-Q. YU, F. QI AND B.-N. GUO, *Estimates for upper and lower bounds of a complete elliptic integral*, Kuàng Yè (Mining) (1995), no. 1, 35–38. (Chinese)
- [48] G.-M. ZHANG, *The upper bounds and lower bounds on Wallis' inequality*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory), **37**, 5 (2007), 111–116. (Chinese)
- [49] D.-J. ZHAO, *On a two-sided inequality involving Wallis' formula*, Shùxué de Shíjiàn yǔ Rènshí (Math. Practice Theory), **34** (2004), 166–168. (Chinese)
- [50] Y.-Q. ZHAO AND Q.-B. WU, *An improvement of the Wallis inequality*, Zhèjiāng Dàxué Xuébào (Lìxué Bǎn) (Journal of Zhejiang University (Science Edition)), **33**, 2 (2006), 372–375. (Chinese)
- [51] Y.-Q. ZHAO AND Q.-B. WU, *Wallis inequality with a parameter*, J. Inequal. Pure Appl. Math., **7**, 2 (2006), Art. 56; Available online at URL: <http://jipam.vu.edu.au/article.php?sid=673>.