

CHARACTERIZATIONS OF THE OPERATOR INEQUALITY $A \geq B \geq C$

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Abstract. Let A , B and C be positive invertible operators on a Hilbert space. Motivated by the characterization of the operator inequality $A \geq B$ due to Fujii, Kamei and Nakamoto in [3], in this paper, we prove the following: $A \geq B \geq C$ if and only if the two operator inequalities

$$A^{r-t} \geq [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}},$$

$$[C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geq C^{r-t}$$

hold for all $p, s \geq 1$, $r \geq t$ and $t \in [0, 1]$.

Finally, characterizations of operator inequalities in terms of the operator equalities are given.

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