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SHARP BOUNDS FOR THE PSI FUNCTION AND HARMONIC NUMBERS

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Abstract. We establish several new bounds for the psi function ψ and harmonic numbers H_n. For example, we prove that

$$\gamma+\frac{1}{2}\log(n^2+n+c)<H_n<\gamma+\frac{1}{2}\log(n^2+n+d),$$

where the constants $c=e^{2(1-\gamma)}-2=0.329302...$ and $d=1/3=0.3333...$ are the best possible, and

$$\gamma+\frac{1}{2}\log\left(\frac{2n+a}{e^{2/(n+1)}-1}\right)<H_n\leqslant\gamma+\frac{1}{2}\log\left(\frac{2n+b}{e^{2/(n+1)}-1}\right),$$

where $a=2$ and $b=e^{2-2\gamma}(e-1)-2=2.0024...$ are the best possible constants. Our estimations give extremely accurate values for γ, and they improve some estimations for γ deduced very recently by C. Mortici.

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