

## A SHARP UNCERTAINTY PRINCIPLE AND HARDY–POINCARÉ INEQUALITIES ON SUB-RIEMANNIAN MANIFOLDS

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*Abstract.* We prove a sharp Heisenberg uncertainty principle inequality and Hardy–Poincaré inequality on the sub-Riemannian manifold  $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  defined by the vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2} \frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2} \frac{\partial}{\partial l}, \quad j = 1, 2, \dots, n$$

where  $|z| = (|x|^2 + |y|^2)^{1/2}$  and  $k \geq 1$ .

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