

QUASI-MONOTONE WEIGHT FUNCTIONS AND THEIR CHARACTERISTICS AND APPLICATIONS

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Abstract. A weight function $w(x)$ on $(0, l)$ or (l, ∞) , is said to be quasi-monotone if $w(x)x^{-a_0} \leq C_0 w(y)y^{-a_0}$ either for all $x \leq y$ or for all $y \leq x$, for some $a_0 \in \mathbb{R}$, $C_0 \geq 1$. In this paper we discuss, complement and unify several results concerning quasi-monotone functions. In particular, some new results concerning the close connection to index numbers and generalized Bary-Stechkin classes are proved and applied. Moreover, some new regularization results are proved and several applications are pointed out, e.g. in interpolation theory, Fourier analysis, Hardy-type inequalities, singular operators and homogenization theory.

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