

WEIGHTED HARDY-TYPE INEQUALITIES IN ORLICZ SPACES

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Abstract. For a given N -function M , and inner and outer weight functions $\omega, e^{-\varphi}$, we obtain Hardy-type inequalities:

$$\int_a^b M(\omega(r)|u(r))e^{-\varphi(r)} dr \leq C \left(\int_a^b M(|u(r)|)e^{-\varphi(r)} dr + \int_a^b M(|u'(r)|)e^{-\varphi(r)} dx \right),$$

holding for every $u \in \mathcal{R}$, where \mathcal{R} is a suitable dilation invariant subset of $W_{loc}^{1,1}(a,b)$, containing $C_0^\infty(a,b)$. The constant C above is independent of u . In many cases considered, the set \mathcal{R} is proven to be maximal possible.

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