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CORACH–PORTA–RECHT INEQUALITY FOR CLOSED RANGE OPERATORS

MARYAM KHOSRAVI

Abstract. By $\mathbb{B}(\mathscr{H})$ we denote the space of all bounded linear operators on a Hilbert space $\mathscr{H}$. In 2001, Seddik characterized all invertible self-adjoint operators using the Corach-Porta-Recht inequality

$$\|SXS^{-1}+S^{-1}XS\| \geqslant 2\|X\|.$$

In this paper, we find a characterization of closed range self-adjoint operators using a version of this inequality for closed range operators.

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1. Introduction and preliminaries

In [1], Corach, Porta and Recht proved that if S is a self-adjoint invertible operator on a Hilbert space $\mathscr{H}$, then for all $X \in \mathbb{B}(\mathscr{H})$ the following (C-P-R) inequality holds:

$$\|SXS^{-1}+S^{-1}XS\| \geqslant 2\|X\|.$$

They used this inequality as a key factor in their study of differential geometry. J. I. Fujii, M. Fujii, Furuta and Nakamoto [2], showed that this inequality is equivalent to Heinz inequality which is one of the most essential inequalities in operator theory.

Seddik [5] could find a characterization of non-zero scalars of invertible self-adjoint operators base on this inequality.

In this paper we discuss about a version of this inequality for Moore-Penrose invertible operators.

DEFINITION 1.1. Let $\mathscr{A}$ be an algebra with involution and $a \in \mathscr{A}$. If there exists an element $x \in \mathscr{A}$ satisfied the following four equations

$$\begin{array}{ll} axa = a & xax = x \\ (ax)^* = ax & (xa)^* = xa, \end{array} \qquad (*)$$

then x is called a Moore-Penrose inverse of a and denoted by $a^\dagger$.

It is easy to show that the Moore-Penrose inverse of an element a is unique.

If $a \in \mathscr{A}$ is Moore-Penrose invertible, then

(a) $a^{\dagger\dagger} = a$,

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(b) a^* is Moore-Penrose invertible and $a^{*\dagger} = a^{\dagger *}$,

(c) If a is invertible, then $a^\dagger = a^{-1}$,

(d) $(aa^*)^\dagger = a^{*\dagger} a^\dagger$.

Harte and Mbekhta in [3] proved that if $\mathscr{H}$ is a Hilbert space and $T \in \mathbb{B}(\mathscr{H})$ then the following conditions are equivalent:

i) T has a generalized inverse (that is there exists an operator $S \in \mathbb{B}(\mathscr{H})$ for which $TST = T$ and $STS = S$).

ii) $\mathscr{R}(T)$ is closed.

iii) T has a Moore-Penrose inverse.

In this case $TT^\dagger$ is the projection on $\mathscr{R}(T)$ and $T^\dagger T$ is the projection on $\mathscr{R}(T^*)$.

In this note, we present a version of C-P-R inequality for Moore-Penrose invertible operators. In addition, for a close subspace $\mathscr{K}$ of Hilbert space $\mathscr{H}$, we give a characterization of all hermitian operators with range $\mathscr{K}$.

2. Main results

In [4], McIntosh proved that

$$\|A^*AX + XBB^*\| \geqslant 2\|AXB\|,$$

for all operators $A, X, B \in \mathbb{B}(\mathscr{H})$. Using this inequality we can state the following result.

THEOREM 2.1. *Let S be a hermitian operator on Hilbert space $\mathscr{H}$ such that $\mathscr{R}(S)$ is closed. Then*

$$\|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

where $P = SS^\dagger$.

Proof. From McIntosh inequality and the relation $SS^\dagger = (SS^\dagger)^* = S^\dagger S$, we have

$$\|SXS^\dagger + S^\dagger XS\| = \|SS(S^\dagger XS^\dagger) + (S^\dagger XS^\dagger)SS\| \geqslant 2\|SS^\dagger XS^\dagger S\| = 2\|PXP\|. \qquad \square$$

In general, it is not true that $(ab)^\dagger = b^\dagger a^\dagger$. However, in polar decomposition of an operator we can deduced the next result.

LEMMA 2.2. *Let S be an operator with close range and $S = U|S|$ be the polar decomposition of S. Then*

$$S^\dagger = |S|^\dagger U^*, \qquad \& \qquad |S|^\dagger = S^\dagger U.$$

Proof. First note that as a result of polar decomposition, we have $\mathscr{R}(|S|) = \mathscr{R}(S^*)$ and therefore is closed. Since the Moore-Penrose inverse is unique, the following relations lead to the first equation:

1. $S(|S|^\dagger U^*)S = U|S|(|S|^\dagger U^*)S = U|S||S|^\dagger|S| = U|S| = S$.
2. $(|S|^\dagger U^*)S(|S|^\dagger U^*) = |S|^\dagger|S||S|^\dagger U^* = |S|^\dagger U^*$.
3. $S(|S|^\dagger U^*) = U|S||S|^\dagger U^*$ which is hermitian, because $|S||S|^\dagger$ is hermitian.
4. $(|S|^\dagger U^* S) = |S|^\dagger|S|$ which is hermitian.

The second equality is proved similarly. $\square$

REMARK 2.3. By the previous lemma it is seen that

$$|S||S|^\dagger = |S|^\dagger|S| = S^\dagger U|S| = S^\dagger S.$$

In addition

$$|S|^\dagger(H) = |S|^\dagger|S|(H) = S^\dagger S(H) = S^\dagger(H) = S^*(H) = |S|(H).$$

So if $S = U|S|$ is the polar decomposition of S, then U is isometry on $|S|^\dagger(H)$.

Using Lemma 2.2, we can deduce the following version of Theorem 2.1, similarly to [2]:

THEOREM 2.4. *Let S,T be operators on Hilbert space $\mathscr{H}$ such that $\mathscr{R}(S)$ and $\mathscr{R}(T)$ are closed. Then*

$$\|S^*XT^\dagger + S^\dagger XT^*\| \geqslant 2\|PXQ\|,$$

where $P = SS^\dagger$ and $Q = T^\dagger T$.

Proof. First we proved the inequality for the case that $T = S$.
Let $S = U|S|$ be the polar decomposition of S. Then

$$\begin{aligned}
\|S^*XS^\dagger + S^\dagger XS^*\| &= \||S|U^*X|S|^\dagger U^* + |S|^\dagger U^*X|S|U^*\| \\
&= \|U(|S|^\dagger X^*U|S| + |S|X^*U|S|^\dagger)\| \\
&= \||S|^\dagger X^*U|S| + |S|X^*U|S|^\dagger\| && \text{(By Remark 2.3)} \\
&\geqslant 2\||S||S|^\dagger X^*U|S||S|^\dagger\| \\
&= 2\||S||S|^\dagger U^*X|S||S|^\dagger\| \\
&= 2\|U^*SS^\dagger XS^\dagger S\| \\
&= 2\|SS^\dagger XS^\dagger S\| = 2\|PXQ\|. && (U^* \text{ is isometry on } S(H))
\end{aligned}$$

Now let S,T be two arbitrary operators for which $\mathscr{R}(S)$ and $\mathscr{R}(T)$ is closed. Using the previous part, for closed range operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ and all operators of the form $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ on the Hilbert space $\mathscr{H} \oplus \mathscr{H}$, we have

$$\left\| \begin{bmatrix} S^* & 0 \\ 0 & T^* \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^\dagger & 0 \\ 0 & T^\dagger \end{bmatrix} + \begin{bmatrix} S^\dagger & 0 \\ 0 & T^\dagger \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & T^* \end{bmatrix} \right\|$$
$$\geqslant 2 \left\| \begin{bmatrix} SS^\dagger & 0 \\ 0 & TT^\dagger \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^\dagger S & 0 \\ 0 & T^\dagger T \end{bmatrix} \right\|,$$

That is

$$\|S^*XT^\dagger + S^\dagger XT^*\| \geqslant 2\|PXQ\|. \qquad \square$$

In [5], Seddik obtained the following characterization of the invertible operators which satisfy the C-P-R inequality for all $X \in \mathbb{B}(\mathscr{H})$.

THEOREM 2.5. [5] *The set of all invertible operators S, for which*

$$\forall X \in \mathscr{H}, \quad \|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|.$$

is the set $\{\lambda M : \lambda \in \mathbb{C}^*, M$ is an invertible self-adjoint operator$\}$.

Now we prove a version of this theorem for Moore-Penrose invertible operators.

THEOREM 2.6. *Let $\mathscr{K}$ be a closed subspace of $\mathscr{H}$ and P be the projection on $\mathscr{K}$. If $S \in \mathbb{B}(\mathscr{H})$ with $\mathscr{R}(S) = \mathscr{R}(S^*) = \mathscr{K}$ and*

$$\forall X \in \mathbb{B}(\mathscr{H}), \quad \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

then $S = \lambda T$, for some non-zero $\lambda \in \mathbb{C}$ and some self-adjoint operator T with $\mathscr{R}(T) = \mathscr{K}$.

Proof. From the hypothesis, we can simply write

$$\|S(PXP)S^\dagger + S^\dagger(PXP)S\| = \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$$

So from Theorem 2.5, we have $S = \lambda T$ as operators on $\mathbb{B}(\mathscr{K})$. Since $S = 0$ on $\mathscr{K}^\perp$, we can get the result. $\square$

From Theorem 2.1 and 2.6, the following theorem is immediately follows.

THEOREM 2.7. *Let $\mathscr{K}$ be a closed subspace of $\mathscr{H}$ and P be the projection on $\mathscr{K}$. If $S \in \mathbb{B}(\mathscr{H})$ with $\mathscr{R}(S) = \mathscr{R}(S^*) = \mathscr{K}$, then the following conditions are equivalent:*

- $\forall X \in \mathbb{B}(\mathscr{H}), \quad \|SXS^\dagger + S^\dagger XS\| \geqslant 2\|PXP\|,$
- *$S = \lambda T$ for some self-adjoint operator T with $\mathscr{R}(T) = \mathscr{K}$, and non-zero scalar λ.*

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