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FUNCTIONAL EQUATIONS AND SHARP WEAK–TYPE INEQUALITIES FOR THE MARTINGALE SQUARE FUNCTION

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Abstract. The paper aims at the identification of the best constants in the weak-type (p,p) inequalities for the martingale square function, $1 \leqslant p < \infty$. To accomplish this, a related optimal stopping problem for the space-time Brownian motion is investigated. Interestingly, the analysis of the cases $1 \leqslant p \leqslant 2$ and $2 < p < \infty$ requires completely different methods. Namely, in the first case the corresponding value function can be written down explicitly; in the second case the approach rests on the careful analysis of an interesting, integral functional equation.

Mathematics subject classification (2010): 60G40, 60G42, 35R35, 45G10.
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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with formulas, graphs and mathematical tables*, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
[2] B. BOLLOBÁS, *Martingale inequalities*, Math. Proc. Cambridge Phil. Soc. **87** (1980), 377–382.
[3] D. L. BURKHOLDER, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.
[4] D. L. BURKHOLDER, *Sharp inequalities for martingales and stochastic integrals*, Colloque Paul Lévy (Palaiseau, 1987), Astérisque **157–158** (1988), 75–94.
[5] D. L. BURKHOLDER, *Explorations in martingale theory and its applications*, École d'Ete de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.
[6] K. E. DAMBIS, *On the decomposition of continuous submartingales*, Theor. Prob. Appl. **10** (1965), 401–410.
[7] B. DAVIS, *On the L^p norms of stochastic integrals and other martingales*, Duke Math. J., **43** (1976), 697–704.
[8] L. DUBINS AND G. SCHWARZ, *On continuous martingales*, Proc. Nat. Acad. Sci. USA **53** (1965), 913–916.
[9] C. DELLACHERIE AND P.-A. MEYER, *Probabilities and potential B: Theory of martingales*, North Holland, Amsterdam, 1982.
[10] A. KHINTCHINE, *Über dyadische Brüche*, Math. Z. **18** (1923), 109–116.
[11] J. E. LITTLEWOOD, *On bounded bilinear forms in an infinite number of variables*, Quart. J. Math. Oxford, **1** (1930), 164–174.
[12] J. MARCINKIEWICZ, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math., **16** (1937), 84–96.
[13] A. A. NOVIKOV, *On moment inequalities for stochastic integrals*, Theory Probab. Appl. **16** (1971), 538–541.
[14] A. OSĘKOWSKI, *Two inequalities for the first moments of a martingale, its square function and its maximal function*, Bull. Pol. Acad. Sci. Math. **53** (2005), no. 4, 441–449.
[15] A. OSĘKOWSKI, *Sharp martingale and semimartingale inequalities*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], 72. Birkhäuser/Springer Basel AG, Basel, 2012.

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[16] R. E. A. C. PALEY, *A remarkable series of orthogonal functions I*, Proc. London Math. Soc., **34** (1932), 241–264.

[17] J. L. PEDERSEN AND G. PESKIR, *Solving non-linear optimal stopping problems by the method of time-change*, Stochastic Anal. Appl., **18** (2000), 811–835.

[18] G. PESKIR, *On the American option problem*, Math. Finance **15** (2005), 169–181.

[19] G. PESKIR, *A change-of-variable formula with local times on curves*, J. Theoret. Probab. **18** (2005), 499–535.

[20] G. PESKIR AND A. SHIRYAEV, *Optimal stopping and free-boundary problems*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006.

[21] D. REVUZ AND M. YOR, *Continuous martingales and Brownian motion*, Third edition, Fundamental Principles of Mathematical Sciences, 293. Springer-Verlag, Berlin, 1999.

[22] L. A. SHEPP, *A first passage time for the Wiener process*, Ann. Math. Statist. **38** (1967), 1912–1914.

[23] G. WANG, *Sharp Square-Function Inequalities for Conditionally Symmetic Martingales*, Trans. Amer. Math. Soc., **328** (1991), no. 1, 393–419.