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GEOMETRIC CONSTANTS AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Abstract. Let X be a real normed space, let Ψ_2 denote the set of all convex functions on $[0,1]$ such that $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$, and let Φ_2 denote the set of all concave function on $[0,1]$ such that $\psi(0)=\psi(1)=1$. For each $\psi \in \Phi_2 \cup \Psi_2$, it is shown that $\| \|x\|^{-1}x + \|y\|^{-1}y\| \leqslant C_\psi \|x-y\| \|(x,y)\|_\psi^{-1}$ for all nonzero vectors x, $y \in X$, where $C_\psi = 4\max\psi(t)$. The case of $\psi = \psi_p$ ($p > 0$), defined as $\psi_p(t) = ((1-t)^p + t^p)^{1/p}$, is due to Al-Rashed, and is due to Dunkl and Williams when $p=1$. In particular, it is shown that for certain $\psi \in \Phi_2$, the inequality holds for $C_\psi = 2\psi(1/2)$ if and only if X is an inner product space; this generalizes the works of Al-Rashed and Kirk-Smiley.

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REFERENCES

[1] A. M. AL-RASHED, *Norm inequalities and characterizations of inner product spaces*, J. Math. Anal. Appl., **176** (1993), 587–593.
[2] G. BIRKHOFF, *Orthogonality in linear metric spaces*, Duke Math. J., **1** (1935), 169–172.
[3] F. F. BONSALL AND J. DUNCAN, *Numerical ranges II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge University Press, Cambridge, 1973.
[4] J. A. CLARKSON, *Uniformly convex spaces*, Trans. Amer. Math. Soc., **40** (1936), 396–414.
[5] M. M. DAY, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc., **62** (1947), 320–337.
[6] C. F. DUNKL AND K. S. WILLIAMS, *A simple norm inequality*, Amer. Math. Monthly, **71** (1964), 53–54.
[7] R. C. JAMES, *Orthogonality in normed linear spaces*, Duke Math. J., **12** (1945), 291–302.
[8] R. C. JAMES, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc., **53** (1947), 559–566.
[9] R. C. JAMES, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., **61** (1947), 265–292.
[10] A. JIMENEZ-MELADO, E. LLORENS-FUSTER, AND E. M. MAZCUNAN-NAVARRO, *The Dunkl–Williams constant, convexity, smoothness and normal structure*, J. Math. Anal. Appl., **342** (2008), 298–310.
[11] W. A. KIRK AND M. F. SMILEY, *Another characterization of inner product spaces*, Amer. Math. Monthly, **71** (1964), 890–891.
[12] E. R. LORCH, *On certain implications which characterize Hilbert space*, Ann. of Math., **49** (1948), 523–532.
[13] K.-S. SAITO, M. KATO AND Y. TAKAHASHI, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl., **244** (2000), 515–532.
[14] Y. TAKAHASHI, M. KATO AND K.-S. SAITO, *Strict convexity of absolute norms on* $\mathbb{C}^2$ *and direct sums of Banach spaces*, J. Inequal. Appl., **7** (2002), 179–186.

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