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WEIGHTED HARDY–TYPE INEQUALITIES ON THE CONE OF QUASI–CONCAVE FUNCTIONS

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Abstract. The paper is devoted to the study of weighted Hardy-type inequalities on the cone of quasi-concave functions, which is equivalent to the study of the boundedness of the Hardy operator between the Lorentz Γ-spaces. For described inequalities we obtain necessary and sufficient conditions to hold for parameters $q \geqslant 1$, $p > 0$ and sufficient conditions for the rest of the range of parameters.

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REFERENCES

[1] M. ARIÑO, B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.
[2] I. U. ASEKRITOVA, N. YA. KRUGLJAK, L. MALIGRANDA, L.-E. PERSSON, *Distribution and rearrangement estimates of the maximal function and interpolation*, Studia Math. **124** (1997), 107–132.
[3] G. BENNETT, K.-G. GROSSE-ERDMANN, *Weighted Hardy inequalities for decreasing sequences and functions*, Math. Ann. **334** (2006), 489–531.
[4] C. BENNETT, R. SHARPLEY, *Interpolation of operators*, Academic Press, Boston, 1988.
[5] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces. An introduction*, Springer, Berlin, 1976.
[6] M. J. CARRO, L. PICK, J. SORIA, V. D. STEPANOV, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), 397–428.
[7] M. J. CARRO, A. GOGATISHVILI, J. MARTIN, L. PICK, *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*, J. Operator Theory **59** (2008), 309–332.
[8] A. GOGATISHVILI, L. PICK, *Discretization and anti-discretization of rearrangement-invariant norms*, Publ. Mat. **47** (2003), 311–358.
[9] A. GOGATISHVILI, V. D. STEPANOV, *Reduction theorems for weighted integral inequalities on the cone of monotone functions*, Russian Math. Surveys **68** (2013), 597–664.
[10] A. GOGATISHVILI, V. D. STEPANOV, *Reduction theorems for operators on the cones of monotone functions*, J. Math. Anal. Appl. **405** (2013), 156–172.
[11] A. GOGATISHVILI, V. D. STEPANOV, *Operators on the cones of monotone functions*, Doklady Math. **86** (2012), 562–565.
[12] M. L. GOLDMAN, *Sharp estimates for the norms of Hardy-type operators on cones of quasimonotone functions*, Proc. Steklov Inst. Math. **232** (2001), 109–137.
[13] M. L. GOLDMAN, *Order-sharp estimates for Hardy-type operators on cones of quasimonotone functions*, Eurasian Math. J. **2**, 3 (2011), 143–146.
[14] M. L. GOLDMAN, H. P. HEINIG, V. D. STEPANOV, *On the principle of duality in Lorentz spaces*, Canad. J. Math. **48** (1996), 959–979.
[15] M. L. GOLDMAN, M. V. SOROKINA, *Three-weighted Hardy-type inequalities on the cone of quasimonotone functions*, Dokl. Math. **71** (2005), 209–213.
[16] H. HEINIG, L. MALIGRANDA, *Weighted inequalities for monotone and concave functions*, Studia Math. **116** (1995), 133–165.

Paper MIA-17-64

[17] M. JOHANSSON, V. D. STEPANOV, E. P. USHAKOVA, *Hardy inequality with three measures on monotone functions*, Math. Inequal. Appl. **11** (2008), 393–413.
[18] L. V. KANTOROVICH, G. P. AKILOV, *Functional Analysis*, Pergamon Press, Oxford, 1982.
[19] A. KUFNER, L.-E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific, New Jersey, 2003.
[20] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy inequality. About its history and some related results*, Vydavatelsky Servis, Pilsen, 2007.
[21] G. G. LORENTZ, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.
[22] L. MALIGRANDA, *Weighted inequalities for quasi-monotone functions*, J. London Math. Soc. **57** (1998), 363–370.
[23] R. OINAROV, *Two-sided estimates for the norm of some classes of integral operators*, Proc. Steklov Inst. Math. **204** (1994), 205–214.
[24] B. OPIC, A. KUFNER, *Hardy-type Inequalities*, Pitman Research Notes in Math., Series 219, Longman Sci & Tech., Harlow, 1990.
[25] L.-E. PERSSON, N. SAMKO, P. WALL, *Quasi-monotone weight functions and their characteristics and applications*, Math. Inequal. Appl. **12** (2012), 685–705.
[26] L.-E. PERSSON, V. D. STEPANOV, E. P. USHAKOVA, *Equivalence of Hardy-type inequalities with general measures on the cones of non-negative respective non-increasing functions*, Proc. Amer. Math. Soc. **134** (2006), 2363–2372.
[27] D. V. PROKHOROV, *Hardy's inequality with three measures*, Proc. Steklov Inst. Math. **255** (2006), 221–233.
[28] D. V. PROKHOROV, *Inequalities of Hardy type for a class of integral operators with measures*, Anal. Math. **33** (2007), 199–225.
[29] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
[30] G. SINNAMON, *Embeddings of concave functions and duals of Lorentz spaces*, Publ. Mat. **46** (2002), 489–515.
[31] G. SINNAMON, V. D. STEPANOV, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. **54** (1996), 89–101.
[32] V. D. STEPANOV, *Integral operators on the cone of monotone functions*, J. London Math. Soc. **48** (1993), 465–487.
[33] V. D. STEPANOV, *The weighted Hardy's inequality for non-increasing functions*, Trans. Amer. Math. Soc. **338** (1993), 173–186.