

## PÓLYA TYPE INTEGRAL INEQUALITIES: ORIGIN, VARIANTS, PROOFS, REFINEMENTS, GENERALIZATIONS, EQUIVALENCES, AND APPLICATIONS

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*Abstract.* In the article, the author gets to the bottom of the origin of Pólya's integral inequality, plots out the development of the theory of inequalities, collects variants and proofs of Pólya's integral inequality, surveys Iyengar-Mahajani's, Agarwal-Dragomir's, Cerone-Dragomir's, and Qi's refinements, generalizations, and applications of Pólya's integral inequality, and find equivalences between these integral inequalities.

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