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INEQUALITIES FOR THE BETA FUNCTION

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Abstract. Let $g(x) := (e/x)^x\Gamma(x+1)$ and $F(x,y) := g(x)g(y)/g(x+y)$. Let $D^{(k)}_{x,y}$ be the kth differential in Taylor's expansion of $\log F(x,y)$. We prove that when $(x,y) \in \mathbb{R}^2_+$ one has $(-1)^{k-1}D^{(k)}_{x,y}(X,Y) > 0$ for every $X,Y \in \mathbb{R}_+$, and that when k is even the conclusion holds for every $X,Y \in \mathbb{R}$ with $(X,Y) \neq (0,0)$. This implies that Taylor's polynomials for $\log F$ provide upper and lower bounds for $\log F$ according to the parity of their degree. The formula connecting the Beta function to the Gamma function shows that the bounds for F are actually bounds for Beta.

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