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GEOMETRICALLY CONVERGENT SEQUENCES OF UPPER AND LOWER BOUNDS ON THE WALLIS RATIO AND RELATED EXPRESSIONS

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Abstract. Sequences of algebraic upper and lower bounds on the Wallis ratio $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ are given with the relative errors that converge to 0 geometrically and uniformly on any interval of the form $[x_0,\infty)$ for $x_0 > -\frac{1}{2}$; moreover, the relative and absolute errors converge to 0 as $x \to \infty$. These conclusions are based on corresponding results for the digamma function $\psi := \Gamma'/\Gamma$. Relations with other relevant results are discussed, as well as the corresponding computational aspects. This work was motivated by studies of exact bounds involving the Student probability distribution.

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