AN EXTENSION OF TURÁN'S INEQUALITY

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Abstract. Let $p_m(x) = P_m^{(\lambda)}(x)/P_m^{(\lambda)}(1)$ be the m-th ultraspherical polynomial normalized by $p_m(1) = 1$. We prove the inequality $|x| p_n^2(x) - p_{n-1}(x) p_{n+1}(x) \geqslant 0$, $x \in [-1,1]$, for $-1/2 < \lambda \leqslant 1/2$. Equality holds only for $x = \pm 1$ and, if n is even, for $x = 0$. Further partial results on an extension of this inequality to normalized Jacobi polynomials are given.

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